THE DIFFERENTIAL EQUATION OF THE ELLIPTIC CYLINDER*

BY

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1. The differential equation of the elliptic cylinder, often called Mathieu's equation, may be written in either of the forms

$$\frac{d^2y}{dz^2} + (a + k^2\cos^2 z)y = 0$$

or

$$\frac{d^2y}{dz^2} + (a + k^2\cosh^2 z)y = 0.$$

It is not capable of reference to the hypergeometric type and so requires methods peculiar to itself. The principal difficulty in the solution of the equation consists in the determination of a number, the characteristic exponent, which appears in the solution. It is known from general considerations of the theory of linear differential equations that for the trigonometric form of the equation, y may be expressed as

$$c_1e^{2\mu iz}\phi(e^{2iz}) + c_2e^{-2\mu iz}\psi(e^{2iz}),$$

the parameters a and k^2 being supposed arbitrary. The functions $\phi(e^{2iz})$, $\psi(e^{2iz})$ are series of positive and negative integral powers of e^{2iz}, and the quantity μ which is a function of a and k^2 is the exponent in question, and its determination necessitates extended developments. If μ has been found the coefficients of the series for $\phi(e^{2iz})$, $\psi(e^{2iz})$ must be investigated. They satisfy a particular difference equation which is not of a simple type.

Of special importance are those cases of the equation for which there exists a periodic solution with period π or 2π. These solutions which are sometimes called Mathieu's functions do not exist in the general case, but require that a should have special values when k^2 is given. The determination of these values is an important problem. It is to be noticed that in these cases the general solution referred to above is no longer valid.† Periodic solutions present themselves in problems of mathematical physics, for example in

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† H. Weber, *Die Partiellen Differential-Gleichungen der Mathematischen Physik*, vol. 2, p. 265. Bruns, Astronomische Nachrichten, No. 2533, p. 193, No. 2553, p. 129.

determining the vibrations of an elliptic membrane, while the solutions which are not periodic find an application in the theory of orbits.

The solution of the problems indicated in this summary has previously been attempted without complete success. For the bibliography reference may be had to Whittaker and Watson, *Modern Analysis*, 3d edition, chapter xix. The most valuable references are to papers by Dougall (*The solution of Mathieu's differential equation*, Proceedings of the Edinburgh Mathematical Society, vol. 34, part 1, p. 176, 1915–16) and Whittaker (*On the functions associated with the elliptic cylinder in harmonic analysis*, Proceedings of the Fifth International Congress of Mathematicians, 1912, Cambridge). Dougall obtains, among other results, an equation for the characteristic exponent but not in an explicit form. Whittaker proves that the periodic solutions satisfy an integral equation which he uses to derive the expressions for some of the earlier cases.

In the present paper the problems stated above are treated more fully than has previously been done. First the general case is considered. The difference equation for the coefficients of the solution of the differential equation is replaced by an infinite system of equations, and their solution given in two different forms. The characteristic exponent μ is shown to satisfy an equation found in two corresponding forms.

Second, the characteristic exponent is studied in detail. The differential equation is replaced by a Riccati equation, a particular solution of which is found, and from this solution an explicit expression for μ is determined. This expression which is valid only within a limited range of values of k^2 leads to a form of the equation for μ valid for all values of k^2. It is found that μ is an analytic function of k^2 when a is given, and the character of the singular points of μ is determined. The convergence of the solution constructed in the first part of the paper is then proved.

In the third place periodic solutions are considered. On the supposition that they exist they are determined in two different forms. Proof is then given of the existence of periodic solutions and a determination of all of them is made. An equation for the parameter a is given to each of whose roots a periodic solution belongs. This equation has no multiple roots if k^2 is real. Two expressions for periodic solutions are given in forms valid for all values of k^2. The parameters a and the solutions are analytic functions of k^2.

Finally some miscellaneous results are collected. The differential equation is shown not to have two periodic solutions for the same values of a and k^2, and if k^2 is real it is shown that each a is a decreasing function of k^2. Some indications as to the determination of the singular points of a are added.

THE GENERAL CASE

2. The equation of the elliptic cylinder may be written in the form

$$d^2y/dz^2 + (\tfrac{1}{2}k^2c^2 \cosh 2z - s^2)y = 0.$$

Then a solution*

$$y = \sum_{-\infty}^{+\infty} c_n e^{(2n+\omega)z}$$

exists where ω and the coefficients c_n are independent of z. Substitution of the solution in the equation gives

$$\{(2n + \omega)^2 - s^2\}c_n + \tfrac{1}{4}k^2c^2(c_{n-1} + c_{n+1}) = 0.$$

If c_n is replaced by $(-1)^n C_n$,

$$C_{n+1} + C_{n-1} = \frac{(n + \frac{1}{2}\omega)^2 - \frac{1}{4}s^2}{\frac{1}{16}k^2c^2} C_n.$$

Put $\frac{1}{2}\omega = \mu$, $\frac{1}{2}s = r$, $\frac{1}{4}kc = \lambda$, and this equation becomes

$$C_{n+1} + C_{n-1} = \frac{(n + \mu)^2 - r^2}{\lambda^2} C_n.$$

Let

$$-\frac{\lambda^2}{(n + \mu)^2 - r^2} = x_n ;$$

then if n varies from $-\infty$ to $+\infty$ there results the system of equations

$$\begin{aligned}
&\cdots\cdots\cdots\cdots \\
x_0C_{-1} + C_0 + x_0C_1 &= 0, \\
x_1C_0 + C_1 + x_1C_2 &= 0, \\
x_2C_1 + C_2 + x_2C_3 &= 0, \\
x_3C_2 + C_3 + x_3C_4 &= 0, \\
&\cdots\cdots\cdots\cdots .
\end{aligned}$$

Any solution of this system of equations which makes $\sum_{-\infty}^{+\infty} c_n e^{(2n+\omega)z}$ convergent furnishes a solution of the differential equation.

3. If it is supposed that $(n+\mu)^2 - r^2 \neq 0$ for every value of n, then the infinite determinant

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* Floquet, Annales de l'Ecole Normale Supérieure, 1883.

$$\begin{vmatrix} 1 & x_i & 0 & 0 & 0 & \cdots \\ x_{i+1} & 1 & x_{i+1} & 0 & 0 & \cdots \\ 0 & x_{i+2} & 1 & x_{i+2} & 0 & \cdots \\ . & . & . & . & . & \cdots \end{vmatrix} = (x_i)$$

is convergent for all values of i. This is true since a sufficient condition of convergence of the infinite determinant

$$\begin{vmatrix} 1+c_{11} & c_{12} & c_{13} & \cdots \\ c_{21} & 1+c_{22} & c_{23} & \cdots \\ . & . & . & \cdots \end{vmatrix}$$

is that the series

$$c_{11} + c_{12} + c_{21} + c_{13} + c_{22} + c_{31} + \cdots$$

be absolutely convergent.* This condition for the determinant (x_i) is that

$$\frac{\lambda^2}{(i+\mu)^2 - r^2} + 2\frac{\lambda^2}{(i+1+\mu)^2 - r^2} + 2\frac{\lambda^2}{(i+2+\mu)^2 - r^2} + \cdots$$

be absolutely convergent. Since the terms of this series are comparable in absolute value with those of the series

$$\frac{1}{i^2} + \frac{1}{(i+1)^2} + \frac{1}{(i+2)^2} + \cdots$$

the condition is satisfied. By expanding in terms of the elements of the first row it may be verified that the determinants (x_i) satisfy the recurrence relation

$$(x_i) = (x_{i+1}) - x_i x_{i+1}(x_{i+2}).$$

To examine the character of the expansion of (x_i), consider the particular case (x_1) and let

$$D_n = \begin{vmatrix} 1 & x_1 & 0 & 0 & . & . & . & . \\ x_2 & 1 & x_2 & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & x_n & 1 \end{vmatrix}$$

be the determinant of the first n rows and columns of (x_1). Then if D_n is expanded in terms of the last row,

$$D_n = D_{n-1} - x_{n-1}x_n D_{n-2}.$$

* Kowalewski, *Einführung in die Determinantentheorie*, p. 372.

From this recurrence formula it is seen that the terms of the expansion of D_n which contain x_n consist of the product $x_{n-1}\,x_n$ multiplied by $-D_{n-2}$, and those terms which do not contain x_n are the terms of D_{n-1}. Now consider the sum

$$S_n = 1 - \sum_i x_i x_{i+1} + \sum_{j>i+1} x_i x_{i+1} x_j x_{j+1} - \sum_{k>j+1,\, j>i+1} x_i x_{i+1} x_j x_{j+1} x_k x_{k+1} + \cdots.$$

Let the subscripts, subject to the inequalities, take the values from 1 to n in every possible way, and let the series be continued as long as it is possible to construct terms in which no subscript is greater than n. Then the terms of S_n which contain x_n consist of the product $x_{n-1}\,x_n$ multiplied by $-S_{n-2}$, and those which do not contain x_n are the terms of S_{n-1}. Moreover $S_1 = D_1$, $S_2 = D_2$, etc. Therefore $S_n = D_n$.

The determinant (x_1) which is infinite has therefore the formal expansion

$$1 - \sum_i x_i x_{i+1} + \sum_{j>i+1} x_i x_{i+1} x_j x_{j+1} - \cdots$$

where the numbers $i, j, \cdots$ are any succession taken from $1, 2, \cdots$ subject to the inequalities. Since this series is convergent it is an integral function of λ of the form

$$1 + a_4\lambda^4 + a_8\lambda^8 + \cdots.$$

Though not relevant to the subject it may be noticed that the number of terms of a given order in D_n is figurate. If $f_{n,r}$ denotes the nth figurate number of the rth order, the number of terms of order $2p$ in D_{2n} is $f_{2n+1-2p,\,p+1}$ and in D_{2n+1} is $f_{2n+2-2p,\,p+1}$, $0 < p \leqq n$. These are included in the statement that in D_n the number of terms of order $2p$ is $f_{n+1-2p,\,p+1}$, $0 < 2p \leqq n$.

4. A solution of the system of equations for C_i (§ 2) such that $C_n \to 0$ as $n \to \infty$ can now be obtained. Consider the first n equations for which $i \geqq 0$:

$$\begin{aligned} x_1 C_0 + C_1 + x_1 C_2 &= 0, \\ x_2 C_1 + \quad C_2 + x_2 C_3 &= 0, \\ \cdots\quad\cdots\quad\cdots\quad\

$$= \left[- x_1 \begin{vmatrix} 1 & x_2 & 0 & \cdot & \cdot & \cdot \\ x_3 & 1 & x_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & x_n & 1 \end{vmatrix} + (-1)^n C_{n+1} x_1 \cdots x_n \right]$$

$$\div \begin{vmatrix} 1 & x_1 & 0 & 0 & \cdot & \cdot & \cdot \\ x_2 & 1 & x_2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & x_n & 1 \end{vmatrix}.$$

The product $x_1 \cdots x_n \to 0$, $n \to \infty$, and the determinant

$$\begin{vmatrix} 1 & x_1 & 0 & 0 & 0 & \cdot & \cdot \\ x_2 & 1 & x_2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & x_n & 1 \end{vmatrix} \to (x_1).$$

Hence if C_{n+1} is bounded

$$C_1 = \frac{-x_1(x_2)}{(x_1)}.$$

Similarly if $C_1 = 1$,

$$C_2 = \frac{-x_2(x_3)}{(x_2)}, \quad \text{etc.}$$

It follows that

$$C_0 : C_1 : C_2 : \cdots : C_n : \cdots$$
$$= (x_1) : -x_1(x_2) : x_1x_2(x_3) : \cdots : (-1)^n x_1 \cdots x_n(x_{n+1}) : \cdots$$

is a system of values of the ratios of C_i satisfying the given system of equations. Since $(x_{n+1}) \to 1$ as $n \to \infty$, and $x_1 \cdots x_n \to 0$ as $n \to \infty$, it follows that $C_n \to 0$ as $n \to \infty$. That the above ratios furnish a solution of the equations may be verified by direct substitution, and by making use of the recurrence relation

$$(x_i) = (x_{i+1}) - x_i x_{i+1}(x_{i+2}).$$

It is then evident that the restriction $C_i \neq 0$ may be removed.

The system of ratios for C_i may be extended to negative values of i by letting C_{-n} correspond to

$$\frac{(-1)^n(x_{-n+1})}{x_{-n+1}\cdots x_{-1}x_0}.$$

This may be verified as before by substitution in the equations. The coefficients $C_{\pm n}$ are meromorphic functions of λ.

In order to obtain a solution of the differential equation, C_{-n} must approach zero. This condition is not fulfilled independently, but depends upon the value of μ, the characteristic exponent. It is evident from the value of C_{-n} that (x_{-n}) must approach zero, $n\to\infty$. The product $x_{-n+1}\cdots x_{-1}x_0$ which occurs in the denominator tends to zero. Hence the numerator must tend to zero more rapidly to satisfy the requirement. Then μ must be a root of the equation

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x_0 & 1 & x_0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_1 & 1 & x_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_2 & 1 & x_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

where the determinant is infinite in both directions.

5. The system of equations for C_n may be transformed with advantage. Put $C_n=f(n)$, and the equation to be transformed is

$$f(n+1)+f(n-1)=\frac{(n+\mu)^2-r^2}{\lambda^2}f(n).$$

If $n+\mu=z$,

$$f(z-\mu+1)+f(z-\mu-1)=\frac{z^2-r^2}{\lambda^2}f(z-\mu),$$

or

$$\phi(z+1)+\phi(z-1)=\frac{z^2-r^2}{\lambda^2}\phi(z),$$

where $f(z-\mu)=\phi(z)$. If

$$\phi(z)=\frac{\lambda^{2z}}{\Pi(z+r)\Pi(z-r)}v_z,\quad \Pi(z)=\Gamma(z+1),$$

there results the equation

$$\frac{\lambda^{2z+2}}{\Pi(z+1+r)\Pi(z+1-r)}v_{z+1}+\frac{\lambda^{2z-2}}{\Pi(z+r-1)\Pi(z-r-1)}v_{z-1}$$
$$=\frac{z^2-r^2}{\lambda^2}\,\frac{\lambda^{2z}}{\Pi(z+r)\Pi(z-r)}v_z,$$

or

$$\frac{\lambda^4}{(z+r)(z+r+1)(z-r)(z-r+1)}v_{z+1}+v_{z-1}=v_z.$$

Changing z into $z+1$ this equation becomes

$$v_z-v_{z+1}=-\frac{\lambda^4}{(z+r+1)(z+r+2)(z-r+1)(z-r+2)}v_{z+2}.$$

If a proper solution of this difference equation is chosen, then

$$C_n=f(n)=\phi(n+\mu)=\frac{\lambda^{2n+2\mu}}{\Pi(n+\mu+r)\Pi(n+\mu-r)}v_{n+\mu}.$$

The difference equation may be written

$$v_{n+\mu}-v_{n+\mu+1}=-\frac{\lambda^4}{(n+\mu+r+1)(n+\mu+r+2)(n+\mu-r+1)(n+\mu-r+2)}v_{n+\mu+2},$$

or if $u_n=v_{n+\mu}=v_z$,

$$u_n-u_{n+1}=-\frac{\lambda^4}{(n+\mu+r+1)(n+\mu+r+2)(n+\mu-r+1)(n+\mu-r+2)}u_{n+2}.$$

Let $n=0, 1, 2, \cdots$, and put y_{n+1} for the coefficient of u_{n+2}; then there results the system of equations

$$\begin{aligned}
-u_0+u_1+y_1u_2&=0,\\
-u_1+u_2+y_2u_3&=0,\\
-u_2+u_3+y_3u_4&=0,\\
&\cdots\cdots\cdots.
\end{aligned}$$

The determinant

$$\begin{vmatrix}
1 & y_1 & 0 & 0 & \cdot & \cdot & \cdot\\
-1 & 1 & y_2 & 0 & \cdot & \cdot & \cdot\\
0 & -1 & 1 & y_3 & 0 & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{vmatrix}$$

is convergent. This may be proved as follows. Let E_n denote the determinant of order n formed with the first n rows and columns of this infinite determinant. Then

$$E_n=1+\sum y_i+\sum y_iy_j+\sum y_iy_jy_k+\cdots$$

where the summations are extended to all products in which no consecutive y's occur, and the series is continued as far as such products can be formed.

This expansion may be proved by induction. By expanding E_{n+2} in terms of the last row, it follows that $E_{n+2}=E_{n+1}+y_{n+1}E_n$. Moreover $E_1=1$, $E_2=1+y_1$, $E_3=1+y_1+y_2$, $E_4=1+y_1+y_2+y_3+y_1y_3$. If the terms of E_n are divided into the classes of those which contain y_{n-1} and those which do not, the rule of formation of the terms of E_n is verified. To prove that E_n approaches a limit as $n\to\infty$, consider the series

$$E = 1 + \sum y_i + \sum y_i y_j + \cdots,$$

where $i, j, \cdots = 1, \cdots, \infty$, $i \neq j$, etc., and no consecutive y's occur in any product. This series is absolutely convergent, for if the absolute values $|y_i|$ are taken and the restriction as to non-consecutive y's removed, the series becomes the expansion of

$$(1 + |y_1|)(1 + |y_2|)\cdots$$

which is a convergent product. This is evident since

$$\sum |y_n| = |\lambda^4| \sum \frac{1}{|(n+\mu+r)(n+\mu+r+1)(n+\mu-r)(n+\mu-r+1)|}$$

is a convergent series. Hence since $|E-E_n|\to 0$, $n\to\infty$, $E_n\to E$ which was to be proved.

6. It is found that a solution of the system of equations for u_n exists such that $u_n\to 1$ as $n\to\infty$. If

$$[y_1] = \begin{vmatrix} 1 & y_1 & 0 & 0 & \cdot & \cdot & \cdot \\ -1 & 1 & y_2 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & 1 & y_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

this solution will be shown to be $u_n=[y_{n+1}]$. Since the determinant $[y_{n+1}]=1+\sum y_i+\cdots$ where $i>n$, and since the element $y_p\to 0$ as $p\to\infty$, then $[y_{n+1}]\to 1$ as $n\to\infty$. By expanding $[y_{n+1}]$ in terms of the first column,

$$-[y_{n+1}] + [y_{n+2}] + y_{n+1}[y_{n+3}] = 0.$$

But this equality is the result of substituting $u_n=[y_{n+1}]$ in the equations for u_n. Hence $u_n=[y_{n+1}]$ is a solution of the equations, and $u_n\to 1$, $n\to\infty$. The ratios of the coefficients C_n are then given for all values of n by the expression

$$C_n = \rho \frac{\lambda^{2n+2\mu}}{\Pi(n+\mu+r)\Pi(n+\mu-r)}[y_{n+1}]$$

where ρ is independent of n. These expressions for C_n are not equal to those given above (§ 4) if $\rho=1$.

It is then evident that μ must be chosen so that $[y_n]\to 0$ as $n\to-\infty$, or μ must be a solution of the equation

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & y_{-1} & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & -1 & 1 & y_0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & -1 & 1 & y_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

It is necessary to prove the convergence of the series assumed as the solution of the differential equation. For convergence μ must satisfy the equation already obtained in two forms. It remains to show that this condition is sufficient for convergence. The proof of this fact is postponed until § 11.

If $(n+\mu)^2-r^2=0$ the form given for the ratio of the coefficients becomes indeterminate but this indetermination may be removed. For if $p+\mu\pm r=0$, $p>0$, the functions

$$\frac{(-1)^n(x_{-n+1})}{x_{-n+1}\cdots x_{-1}x_0}, \quad (-1)^n x_1\cdots x_n(x_{n+1})$$

may be multiplied throughout by $p+\mu\pm r$ which gives a set of finite values when $p+\mu\pm r=0$. If $p+\mu\pm r=0$, $p\leqq 0$, the numerator and denominator of

$$\frac{(-1)^n(x_{-n+1})}{x_{-n+1}\cdots x_{-1}x_0}$$

may be multiplied by $p+\mu\pm r$ which gives finite values when $p+\mu\pm r=0$. The equations for μ will be replaced in the next part of this paper by an equation the terms of which are determinate if $p+\mu\pm r=0$.

The characteristic exponent

7. The exponent μ is the root of a transcendental equation which has been given in what precedes in two different forms. The determination of μ can however be effected in a different manner, which allows the construction of an explicit expression of at least a branch of μ. It will be proved that μ is an analytic function with multiple determination of the principal parameter k^2 of the differential equation. It is found simpler to take the differential equation in the form

$$\frac{d^2y}{dz^2} = (\lambda \cos^2 z + h^2)y, \quad \lambda = -k^2, \quad h^2 = -a.$$

Regarding h as a constant and λ as a variable, a branch of μ will be found as a convergent series of powers of λ, provided h is not equal to certain exceptional values which will be determined. When h has these exceptional values the proper expression for μ will likewise be found.

Whatever h, the solution of the differential equation will first be proved to be an integral function of z and λ, expressed as an absolutely convergent series of powers in each. This follows from the determination of the coefficients of a series of powers in z satisfying the differential equation. For if $y=a_0+a_1z+\cdots$ where a_0 and a_1 are arbitrary, the remaining coefficients $a_2, \cdots$ are determined by comparison from

$$2a_2 + 6a_3z + \cdots = (\lambda \cos^2 z + h^2)(a_0 + a_1z + a_2z^2 + \cdots).$$

Hence

$$\begin{aligned} 2a_2 &= h^2a_0, \\ 6a_3 &= h^2a_1, \\ 12a_4 &= h^2a_2 + \lambda a_0, \\ &\cdots\cdots\cdots. \end{aligned}$$

The series for y determined in this way is known to be an integral function of z for all values of λ, h^2, a_0, a_1. The coefficients a_n are polynomials in λ whose coefficients depend upon a_0, a_1, h^2, and upon the coefficients of the expansion of $\cos^2 z$. If a comparison is made with the differential equation

$$\frac{d^2\bar{y}}{dz^2} = [\lambda(\cos iz)^2 + |h^2|]\bar{y},$$

and if initial values $|a_0|$, $|a_1|$ are taken, a series will be found for $\bar{y}$ whose coefficients are polynomials in λ, which are determined in a manner similar to those above. These polynomials in λ have positive coefficients whose values are greater respectively than the absolute values of the corresponding coefficients in the determination of y. But the series for $\bar{y}$ is likewise an integral function of z. Hence the series for y is one whose terms admit arbitrary redistribution whatever z, λ, a_0, a_1, h^2, and so is an integral function of z and λ.

Let $y_1=f_1(z)$, $y_2=f_2(z)$ be two independent solutions of the differential equation. Then if

$$\begin{aligned} y_1' &= f_1(z+\pi), & y_2' &= f_2(z+\pi), \\ y_1' &= a_{11}y_1 + a_{12}y_2, & y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$$

A solution* $Y_1 = c_1y_1 + c_2y_2$ may be found such that

$$Y_1' = \nu Y_1 \qquad \text{where} \qquad Y_1' = c_1f_1(z+\pi) + c_2f_2(z+\pi).$$

The quantity ν is a root of the equation

$$\begin{vmatrix} a_{11} - \nu & a_{21} \\ a_{12} & a_{22} - \nu \end{vmatrix} = \nu^2 - (a_{11} + a_{22})\nu + a_{11}a_{22} - a_{12}a_{21} = 0.$$

But since the general solution of the differential equation is of the form

$$y_1 = c_1e^{\mu z}\phi(e^{2zi}) + c_2e^{-\mu z}\psi(e^{2zi}),$$

the values of ν must be $e^{\mu\pi}$ and $e^{-\mu\pi}$, so that their product is unity. Hence $a_{11}a_{22} - a_{12}a_{21} = 1$. Therefore ν is a root of the equation

$$\nu^2 - (a_{11} + a_{22})\nu + 1 = 0, \qquad \text{or} \qquad 2\nu = a_{11} + a_{22} \pm [(a_{11} + a_{22})^2 - 4]^{1/2}.$$

The solution $Y_1 = c_1y_1 + c_2y_2$ which satisfies the condition $Y_1' = \nu Y_1$ is then given by $Y_1 = a_{21}y_1 + (\nu - a_{11})y_2$. The solution $Y_2 = a_{21}y_1 + (1/\nu - a_{11})y_2$ also satisfies the similar condition $Y_2' = (1/\nu)Y_2$. If $a_{21} = 0$, then

$$Y_1 = y_2, \quad Y_2 = (a_{11} - a_{22})y_1 + a_{12}y_2$$

are solutions satisfying $Y_1' = a_{22}y_1$, $Y_2' = a_{11}y_2$.

The coefficients a_{11}, a_{12}, a_{21}, a_{22} are functions of λ determined by the two equations

$$f_1(z+\pi) = a_{11}f_1(z) + a_{12}f_2(z),$$
$$f_2(z+\pi) = a_{21}f_1(z) + a_{22}f_2(z).$$

If

$$f_1(z) = a_0 + a_1z + \cdots,$$
$$f_2(z) = b_0 + b_1z + \cdots,$$
$$f_1(z+\pi) = A_0 + A_1z + \cdots,$$
$$f_2(z+\pi) = B_0 + B_1z + \cdots,$$

then

$$A_0 = a_{11}a_0 + a_{12}b_0,$$
$$B_0 = a_{21}a_0 + a_{22}b_0,$$
$$A_1 = a_{11}a_1 + a_{12}b_1,$$
$$B_1 = a_{21}a_1 + a_{22}b_1,$$

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* Goursat, *Analyse Mathématique*, 2d edition, vol. 2, p. 470.

and

$$a_{11} = \frac{A_0b_1 - A_1b_0}{a_0b_1 - a_1b_0}, \qquad a_{12} = \frac{-A_0a_1 + A_1a_0}{a_0b_1 - a_1b_0},$$

$$a_{21} = \frac{B_0b_1 - B_1b_0}{a_0b_1 - a_1b_0}, \qquad a_{22} = \frac{-B_0a_1 + B_1a_0}{a_0b_1 - a_1b_0}.$$

$a_0b_1 - a_1b_0 \neq 0$, since the solutions f_1 and f_2 are independent. Hence a_{11}, a_{12}, a_{21}, a_{22} are integral functions of λ. Since $2\nu = a_{11} + a_{22} \pm [(a_{11} + a_{22})^2 - 4]^{1/2}$, the value of ν may be written

$$2\nu = P(\lambda) \pm [(a_{11}^0 + a_{22}^0)^2 - 4 + C_1\lambda + C_2\lambda^2 + \cdots]^{1/2}$$

where a_{11}^0, a_{22}^0 are the values of a_{11}, a_{22} when $\lambda = 0$. If $(a^0{}_1 + a_{22}^0)^2 - 4 \neq 0$ the radical may be expanded in powers of λ, and $2\nu = P(\lambda) \pm P_1(\lambda)$. The series $P_1(\lambda)$ is convergent if

$$\left| \frac{C_1\lambda + C_2\lambda^2 + \cdots}{(a_{11}^0 + a)_{22}^0 - 4} \right| < 1,$$

and a quantity σ may be determined such that this condition is fulfilled if $|\lambda| < \sigma$. If $\lambda = 0$ the solution of the differential equation is $y = c_1e^{hz} + c_2e^{-hz}$, and the values of ν are $e^{h\pi}$, $e^{-h\pi}$. Hence

$$a_{11}^0 + a_{22}^0 = e^{h\pi} + e^{-h\pi} = 2\cosh h\pi.$$

If $(a_{11}^0 + a_{22}^0)^2 - 4 = 0$, then $e^{h\pi} = \pm 1$, or $h\pi = 2n\pi i$, or $(2n+1)\pi i$. These give exceptional values of h, which will for the present be excluded from the discussion.

The solutions

$$Y_1 = a_{21}y_1 + (\nu - a_{11})y_2,$$
$$Y_2 = a_{21}y_1 + (1/\nu - a_{11})y_2, \qquad a_{21} \neq 0,$$

must reduce if $\lambda = 0$ to multiples of e^{hz} and e^{-hz}. The solutions Y_1 and Y_2 do not both vanish identically if $\lambda = 0$, for since y_1, y_2 do not both vanish identically this would require that $a_{21} = 0$, $\nu - a_{11} = 0$, $1/\nu - a_{11} = 0$. But the last two are contradictory, since $\nu \neq 1/\nu$ unless h is an exceptional value. Hence

$$Y_1 = m_1e^{hz} + \lambda P_1(z,\lambda),$$
$$Y_2 = m_2e^{-hz} + \lambda P_2(z,\lambda),$$

where m_1 and m_2 are independent of z and λ, and m_1 and m_2 are not both equal to zero. If $m_1 \neq 0$, divide by m_1 and there results a solution of the differential equation

$$y = e^{hz} + \lambda \frac{P_1(z,\lambda)}{m_1} = e^{hz} + \lambda Q_1(z,\lambda).$$

Let $Z = y'/y$ where y' denotes differentiation. Then

$$Z = \frac{he^{hz} + \lambda Q_1'(z,\lambda)}{e^{hz} + \lambda Q_1(z,\lambda)} = \frac{e^{-hz}[he^{hz} + \lambda Q_1'(z,\lambda)]}{1 + \lambda e^{-hz} Q_1(z,\lambda)}.$$

Let ρ be an arbitrary quantity; then $\tau < \sigma$ can now be determined such that

$$|\lambda e^{-hz} Q_1(z,\lambda)| < 1 \quad \text{when} \quad |z| < \rho, \quad |\lambda| < \tau.$$

Hence if $|z| < \rho$, $|\lambda| < \tau$, the fraction for Z may be expanded in powers of λ, or $Z = h + z_1\lambda + z_2\lambda^2 + \cdots$. To verify this statement it will be noted that since the series for a_{21}, ν, a_{11}, y_1, y_2 are all absolutely convergent, if $|\lambda| < \sigma$ and $|z| < \rho$, the expressions for Y_1, Y_2 are absolutely convergent series for the same values of λ and z. That is, if $Y_1 = \sum\sum a_{mn} z^m \lambda^n$, $\sum\sum |a_{mn}| z^m \lambda^n$ is a convergent series, so that an arbitrary arrangement of terms is admissible. Under the same conditions the series for $1 + \lambda e^{-hz} Q_1(z, \lambda)$ is absolutely convergent because

$$1 + \lambda e^{-hz} Q_1(z,\lambda) = e^{-hz}[e^{hz} + \lambda Q_1(z,\lambda)],$$

and both factors of this product are absolutely convergent. Finally the series for $[1 + \lambda e^{-hz} Q_1(z, \lambda)]^{-1}$ is absolutely convergent if $|\lambda| < \tau$ where τ is determined as follows. In $Q_1(z, \lambda)$ let all of the coefficients be replaced by their absolute values and let the resulting series be denoted by $\bar{Q}_1(z, \lambda)$. Then determine $\tau < \sigma$ so that $\tau \cdot e^{|\lambda|\rho} \bar{Q}_1(\rho, \tau) < 1$. Then evidently

$$|\lambda e^{-hz} Q_1(z,\lambda)| < 1 \quad \text{if} \quad |\lambda| < \tau, \quad |z| < \rho.$$

The series

$$1 + \lambda e^{|\lambda| z} \bar{Q}_1(z,\lambda) + \lambda^2 e^{2|\lambda| z} \bar{Q}_1^2(z,\lambda) + \cdots$$

is convergent, and the terms in the expansion of $\lambda^n e^{n|\lambda| z} Q_1^n(z, \lambda)$ have positive coefficients. Hence finally the series

$$1 - he^{-hz} Q_1(z,\lambda) + \lambda^2 e^{-2hz} Q_1(z,\lambda) - \cdots$$

is convergent and admits arbitrary rearrangement of terms.

8. If $Z = y'/y$, Z satisfies the Riccati equation $Z' + Z^2 = \lambda \cos^2 z + h^2$, and a solution Z of this equation has been found. If this solution is $Z = h + Z_1\lambda + Z_2\lambda^2 + \cdots$, the coefficients Z_1, Z_2, etc. may now be determined by substitution in the differential equation, and by comparison of coefficients of like powers of λ. There results the system of equations

$$
\begin{aligned}
&Z_1' + 2hZ_1 = \cos^2 z,\\
&Z_2' + 2hZ_2 + Z_1^2 = 0,\\
&Z_3' + 2hZ_3 + 2Z_1Z_2 = 0,\\
&Z_4' + 2hZ_4 + Z_2^2 + 2Z_1Z_3 = 0,\\
&Z_5' + 2hZ_5 + 2Z_1Z_4 + 2Z_1Z_3 = 0,\\
&\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
$$

This is a system of equations which may be solved, the complementary function in each case being ce^{-2hz}. But the solution Z is a periodic function of z with period π. Then the coefficients Z_1 and Z_2 must be periodic, and since the particular solutions of the equations for Z are periodic, the complementary functions must in every case be omitted. It is found that

$$Z_1 = \frac{1}{2^2h} + \frac{\sin 2z + h\cos 2z}{2^2(1+h^2)},$$

and in general Z_n equals a finite series of sines and cosines of even multiples up to $2nz$, i.e.

$$Z_n = b_{n,1} + b_{n,2}\sin 2z + b_{n,3}\cos 2z + \cdots + b_{n,2n}\sin 2nz + b_{n,2n+1}\cos 2nz,$$

where the coefficients b_{ij} are rational functions of h.

9. The general solution of the differential equation

$$\frac{d^2y}{dz^2} = (\lambda\cos^2 z + h^2)y$$

may be taken in the form

$$y = c_1e^{2\mu iz}\phi(e^{2zi}) + c_2e^{-2\mu iz}\psi(e^{2zi}),$$

where ϕ and ψ are series of positive and negative powers of e^{2zi}. It is evident that $\nu = e^{2\mu\pi i}$, or $e^{-2\mu\pi i}$. The Riccati equation $Z'+Z^2=\lambda\cos^2 z+h^2$ is satisfied by y'/y whatever c_1 and c_2. If $c_1=0$, or $c_2=0$, y'/y is a periodic function of z of period π and in no other case. Hence the function Z of the preceding discussion must correspond to one of these cases, and since the sign of μ has not been fixed, it may be supposed that it corresponds to $c_2=0$. Hence $Z=2\mu i+\phi'/\phi$, where ϕ' denotes differentiation with respect to z. Therefore

$$2\mu i + \phi'/\phi = h + Z_1\lambda + Z_2\lambda^2 + \cdots.$$

The series on the right has been determined and is a uniformly convergent series of functions of z if $|\lambda|<\tau$, $|z|<\rho$. Since ρ is arbitrary it may be supposed that $\rho>\pi$. Integration with respect to z from 0 to π gives

$$2\pi\mu i + \int_0^\pi \frac{\phi'}{\phi} dz = \pi h + \lambda \int_0^\pi Z_1 dz + \cdots .$$

The value of

$$\int_0^\pi Z_n dz = \pi b_{n,1}$$

since the integrals of the trigonometric terms vanish. Hence

$$2\mu i + \frac{1}{\pi} \int_0^\pi \frac{\phi'}{\phi} dz = h + \sum_{n=1\cdots\infty} b_{n,1}\lambda^n .$$

But

$$\int_0^\pi \frac{\phi'}{\phi} dz = \frac{1}{\pi} \log \phi \Big]_0^\pi = \frac{\log 1}{\pi} = 2ni,$$

since ϕ has the period π. Finally

$$2\mu i + 2ni = h + \sum_{n=1\cdots\infty} b_{n,1}\lambda^n ,$$

a series for μ convergent if $|\lambda| < \tau$. The integer n is in the nature of the case not determined, since in the form of the general solution of the differential equation $2\mu i$ may be increased by $2ni$.

A similar series for μ will be found if the differential equation is taken in the hyperbolic form

$$\frac{d^2y}{dz^2} = -(h^2 + \lambda \cosh^2 z)y .$$

Substitute $z = iz'$ and the equation becomes

$$\frac{d^2y}{dz'^2} = (h^2 + \lambda \cos^2 z')y .$$

This equation shows that a series giving μ for the hyperbolic form may be derived from the series which gives μ for the trigonometric form by changing the sign of λ and replacing h^2 by $-h^2$. For the trigonometric form of the equation μ is given by the series

$$2i(\mu + n) = h + \frac{1}{2^2 h}\lambda - \frac{3h^2 + 2}{2^6 h^3(1 + h^2)}\lambda^2 + \frac{5h^6 + 10h^4 + 7h^2 + 2}{28h^5(1 + h^2)^3}\lambda^3$$

$$- \frac{175h^{12} + 1750h^{10} + 6027h^8 + 8876h^6 + 7048h^4 + 3264h^2 + 640}{2^{14}h^7(1 + h^2)^4(4 + h^2)^2}\lambda^4$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot .$$

For the hyperbolic form of the equation μ is given by the series

$$2i(\mu + n) = ih - \frac{1}{2^2 ih}\lambda + \frac{-3h^2 + 2}{2^6 ih^3(1 - h^2)}\lambda^2 + \cdots .$$

It is evident that within the range of convergence of these series the value of μ is real for the hyperbolic case and a pure imaginary for the trigonometric.

If h has one of the exceptional values 0, $\pm ni$ the equations for Z_i take a different form. Suppose $h=0$; then Y_1, Y_2 are series of powers of $\lambda^{1/2}$ (§ 7) and

$$Z = z_1\lambda^{1/2} + z_2\lambda + \cdots .$$

The differential equation $Z' + Z^2 = \lambda \cos^2 z$ gives the system

$$\begin{aligned} Z_1' &= 0, \\ Z_2' + Z_1^2 &= \cos^2 z, \\ Z_3' + 2Z_1Z_2 &= 0, \\ Z_4' + Z_2^2 + 2Z_1Z_3 &= 0, \\ &\cdots \end{aligned}$$

These equations must have a periodic solution. The first equation gives $Z_1 = c$, and substituting in the second it is seen that

$$Z_2 = \left(-c^2 + \frac{1}{2}\right)z + \frac{\sin 2z}{4} + c' ;$$

hence $c = 1/2^{1/2}$. Similarly

$$Z_3 = \frac{2^{1/2}\cos 2z}{8} - 2^{1/2}c'z + c'',$$

and $c' = 0$. If this value of Z_3 is substituted in the next equation

$$Z_4 = -\left(\frac{1}{32} + 2^{1/2}c''\right)z + \frac{\sin 4z}{128} - \frac{\sin 2z}{8} + c'''.$$

Hence $c'' = -1/(2^{1/2} \cdot 32)$, etc. Finally

$$Z = \frac{\lambda^{1/2}}{2^{1/2}} + \frac{\sin 2z}{4}\lambda + \left(\frac{2^{1/2}\cos 2z}{8} - \frac{1}{2^{1/2} \cdot 32}\right)\lambda^{3/2} + \left(\frac{\sin 4z}{128} - \frac{\sin 2z}{8}\right)\lambda^2 + \cdots .$$

The value of μ may be found as before and is given by

$$2\pi i(\mu + n) = \frac{\lambda^{1/2}}{2^{1/2}} - \frac{\lambda^{3/2}}{2^{1/2} \cdot 32} + \cdots .$$

When h has one of the other exceptional values it becomes necessary to have recourse to the equation given hereafter for μ(§ 10), the coefficients of which become indeterminate for the exceptional values, but the indetermination can be removed in the usual way.

10. It will now be proved that the exponent μ may be determined by an equation involving a series convergent for all values of λ. It has been shown (§ 7) that $u_n = [y_{n+1}]$ satisfies the equation

$$u_n - u_{n+1} = y_{n+1}u_{n+2}.$$

If $u_n = v_{n+\mu} = v_z$ then

$$v_z - v_{z+1} = -\frac{\lambda^4}{(z+r+1)(z+r+2)(z-r+1)(z-r+2)} v_{z+2}.$$

If

$$\phi(z) = \frac{\lambda^{2z}}{\Pi(z+r)\Pi(z-r)} v_z ,$$

then $\phi(z)$ satisfies the difference equation

$$\phi(z+1) + \phi(z-1) = \frac{z^2 - r^2}{\lambda^2} \phi(z).$$

If

$$f(z) = \phi(z)\phi(-z-1) - \phi(-z)\phi(z+1),$$

then it may be shown that $f(z) = f(z-1) = f(-z)$. This is evident since $\phi(z)$ satisfies the equation above. If the sign of z is changed, $\phi(z)$ satisfies the equation

$$\phi(-z+1) + \phi(-z-1) = \frac{z^2 - r^2}{\lambda^2} \phi(-z).$$

If $(z^2 - r^2)/\lambda^2$ is eliminated the eliminant is

$$\phi(z)\phi(-z-1) - \phi(-z)\phi(z+1) = \phi(z-1)\phi(-z) - \phi(-z+1)\phi(z),$$

or $f(z) = f(z-1)$. It is evident at once that $f(z) = f(-z)$. If z is replaced in turn by μ and by $n+\mu$ it follows that

$$\phi(\mu)\phi(-\mu-1) - \phi(-\mu)\phi(\mu+1) = \phi(n+\mu)\phi(-n-\mu-1) - \phi(-n-\mu)\phi(n+\mu+1).$$

If μ is a solution of the equations (§§ 4, 6)

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_0 & 1 & x_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_1 & 1 & x_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & -1 & 1 & y_0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & -1 & 1 & y_1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

the quantities $\phi(n+\mu)$, $\phi(-n-\mu)\to 0$ as $n\to\infty$. This is evident for $\phi(n+\mu)$. It is also true for $\phi(-n-\mu)$, because from the form of the equations for μ it is seen that if μ is a root of the equations, $-\mu$ is also a root. μ is in fact a value such that $\phi(n-\mu)\to 0$, $\phi(-n-\mu)\to 0$ when $n\to\infty$. The function $f(\mu)$ must then equal zero, or μ satisfies the equation

$$\phi(\mu)\phi(-\mu-1)-\phi(-\mu)\phi(\mu+1)=0.$$

This equation in μ if transformed by the introduction of the function v_z takes a simple form. For since

$$\phi(z)=\frac{\lambda^{2z}}{\pi(z+r)\pi(z-r)}v_z,$$

$$(z)=\lim_{n\to-\infty}\left\{\frac{\lambda^{-2}v_{n+z}v_{-n-z-1}}{\Pi(n+z+r)\Pi(n+z-r)\Pi(-n-z-1+r)\Pi(-n-z-1-r)}\right.$$
$$\left.-\frac{\lambda^{2}v_{-n-z}v_{n+z+1}}{\Pi(-n-z+r)\Pi(-n-z-r)\Pi(n+z+1+r)\Pi(n+z+1-r)}\right\}.$$

Now if $n\to-\infty$, v_{-n-z} and $v_{-n-z-1}\to 1$, and

$$v_{n+z}\to\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & y_{-1} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & -1 & 1 & y_0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & -1 & 1 & y_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

the infinite determinant of § 6. Moreover, since

$$\Pi(z)\Pi(-z-1)=\frac{-\pi}{\sin\pi z},$$

then

$$\frac{\lambda^{-2}}{\Pi(n+z+r)\Pi(n+z-r)\Pi(-n-z-1+r)\Pi(-n-z-1-r)} = \frac{\sin\pi(z+r)\sin\pi(z-r)}{\pi^2\lambda^2},$$

and

$$\frac{\lambda^{2}}{\Pi(-n-z+r)\Pi(-n-z-r)\Pi(n+z+1+r)\Pi(n+z-1-r)} = \frac{\lambda^{2}}{\pi^2(n+z+1+r)(n+z+r)(n+z+1-r)(n+z-r)}.$$

If these results are substituted in the expression for $f(z)$ it follows that

$$f(z) = \frac{\sin\pi(z+r)\sin\pi(z-r)}{\pi^2\lambda^2} \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & y_{-1} & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & -1 & 1 & y_0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & -1 & 1 & y_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

The determinant may be developed in a series of powers of λ, the coefficients of which are of the form given by the construction of $[y_n]$. The equation for μ takes the form

$$(1 + a_4\lambda^4 + a_8\lambda^8 + \cdots)\sin\pi(\mu+r)\sin\pi(\mu-r) = 0.$$

This may be written $\cos 2\pi\mu = \cos 2\pi r - 2b_4\lambda^4 + 2b_8\lambda^8 - \cdots$, where $b_4 = -a_4 \sin\pi(\mu+r)\sin\pi(\mu-r)$, and similarly for $b_8, \cdots$.

The coefficients $b_4, \cdots$ are independent of μ. The coefficients a_{4q} may be written

$$(-1)^q \sum_{p_1=-\infty}^{+\infty} \cdots \sum_{p_q=2}^{\infty} y_{p_1}x_{p_1+p_2}\cdots y_{p_1+p_2+\cdots+p_q}.$$

The function of μ under the summation sign is a rational function of μ having $4q$ poles for $2q$ of which $\mu+r$ is an integer, and for the other $2q$, $\mu-r$ is an integer. If a residue at a pole for which $\mu-r$ is an integer is d, the residue at a pole for which $\mu+r$ is an integer is $-d$, since a_{4q} is periodic with period unity. Hence

$$a_{4q} = (-1)^q\pi[\cot\pi(\mu-r) - \cot\pi(\mu+r)]d + C,$$

where C is independent of μ. If $\mu = il$, then as $l\to\infty$, $a_{4q}\to 0$, $\cot\pi(\mu-r) - \cot\pi(\mu+r)\to 0$, so $C=0$. Hence

$$a_{4q} = \frac{(-1)^q \sin 2r\pi}{\sin \pi(\mu + r) \sin \pi(\mu - r)} d,$$

or $b_{4q} = (-1)^q d \sin 2r\pi$, which is independent of μ.

The form of the coefficients b may be found more precisely as follows. Let $\mu = r + c_1\lambda^4 + \cdots = r + w$, where the coefficients are those of the branch of μ previously determined (§ 9); then

$$\begin{aligned}\cos 2\pi(r + w) &= \cos 2\pi r \cos 2\pi w - \sin 2\pi r \sin 2\pi w \\ &= \cos 2\pi r \left\{1 - \frac{(2\pi w)^2}{2} + \cdots\right\} \\ &\quad - \sin 2\pi r \left\{2\pi w - \frac{(2\pi w)^3}{6} + \cdots\right\}.\end{aligned}$$

If w is replaced by its value as a series in λ the coefficients b may be determined, and they are found to be of the form

$$b_{4q} = C_q(r) \cos 2\pi r + S_q(r) \sin 2\pi r$$

where $C_q(r)$, $S_q(r)$ are rational functions of r. Hence finally

$$\cos 2\pi\mu = \cos 2\pi r + \sum_{q=1}^{\infty} (-1)^q \{C_q(r) \cos 2\pi r + S_q(r) \sin 2\pi r\} \lambda^{4q}.$$

This equation is valid for all values of λ.

If r has one of the exceptional values, viz. if $2r = 0, \pm n$ the form of the functions C_q, S_q must be modified. The modification is found by evaluating the coefficients $C_q(r)$, $S_q(r)$ which are indeterminate. If $r = 0$ the corresponding branch of μ which is found in § 8 may be substituted in the equation for μ giving in another way the result for this case.

11. The proof of the convergence of the series $e^{2\mu z} \sum_{-\infty}^{+\infty} c_n e^{2nz}$ which was postponed in § 6 may now be given. Consider the terms of the series $\sum_1^{\infty} c_n e^{2nz}$. The ratio of the $(n+1)$st term to the nth gives

$$\frac{c_{n+1} e^{2(n+1)z}}{c_n e^{2nz}} = \frac{x_{n+1}(x_{n+2}) e^{2z}}{(x_{n+1})}$$

(see § 4). From the expansion of (x_n) in a series

$$(x_n) = 1 - \sum_i x_i x_{i+1} + \sum_{j>i+1} x_i x_{i+1} x_j x_{j+1} - \cdots$$

and from the value of x_i,

$$x_i = \frac{-\lambda^2}{(n + \mu)^2 - r^2},$$

it is evident that $(x_n)\to 1$, $n\to\infty$. Hence

$$\frac{c_{n+1}}{c_n}e^{2z}=\frac{\lambda^2}{(n+1+\mu)^2-r^2}e^{2z}(1+\epsilon)\to 0,\ n\to\infty,$$

for every value of z. Therefore the part of the series where n is positive is convergent.

The series $\sum_0^\infty c_{-n}e^{-2nz}$ is likewise convergent, μ being a root of the equation

$$\phi(\mu)\phi(-\mu-1)-\phi(-\mu)\phi(\mu+1)=\phi(n-\mu)\phi(-n+\mu-1)$$
$$-\phi(-n+\mu)\phi(n-\mu+1)=0;$$

because the ratio

$$\frac{c_{-n-1}}{c_{-n}}e^{-2z}=\frac{\phi(-n+\mu-1)}{\phi(-n+\mu)}e^{-2z}=\frac{\phi(n-\mu+1)}{\phi(n-\mu)}e^{-2z}$$
$$=\frac{\lambda^2}{(n+1-\mu)^2-r^2}e^{-2z}(1+\epsilon)\to 0,\ n\to\infty.$$

Hence the series $e^{2\mu z}\sum_{-\infty}^{\infty}c_ne^{2nz}$ is convergent for all values of z and therefore satisfies the differential equation.

12. The form of the equation

$$\cos 2\pi\mu=\cos 2\pi r+\sum_{n=1}^{\infty}(-1)^n\{C_n(r)\cos 2\pi r+S_n(r)\sin 2\pi r\}\lambda^{4n}$$

permits the determination of the character of the singular points of μ regarded as a function of λ. The singular points will evidently be those values of λ which make the right hand side equal to ± 1. Let λ_0 be such a value. Then the equation becomes

$$\cos 2\pi\mu=\pm 1-a_1(\lambda-\lambda_0)+\cdots,$$

or taking the plus sign

$$2\sin^2\pi\mu=a_1(\lambda-\lambda_0)+\cdots.$$

Therefore $\sin\pi\mu=(a_1/2)^{1/2}(\lambda-\lambda_0)^{1/2}[1+b_1(\lambda-\lambda_0)+\cdots]$. By inversion of series

$$\mu=c_1(\lambda-\lambda_0)^{1/2}+c_3(\lambda-\lambda_0)^{3/2}+\cdots.$$

Hence the point λ_0 has the character of a critical point. If $\cos 2\pi\mu=-1-a_1(\lambda-\lambda_0)+\cdots$ a similar result follows.

THE PERIODIC SOLUTIONS

13. That the differential equation of the elliptic cylinder has solutions in certain cases which are not of the form which has been supposed as the general one is usually assumed. A proof of the existence of these solutions is given in § 17. These solutions occur when the parameters λ and r have particular relations to each other, and there exists a solution which is periodic with period πi or $2\pi i$. Taking the equation in the trigonometric form it will be written in what follows as

$$\frac{d^2y}{dz^2} + (a + k^2 \cos^2 z)y = 0.$$

The problem is to determine a in terms of k so that the equation has a solution which has the period π or 2π. It was proved by Whittaker that the even periodic functions satisfy the integral equation

$$y(z) + \lambda \int_{-\pi}^{+\pi} e^{k \cos z \cos \theta} y(\theta) d\theta = 0,$$

that the odd periodic functions satisfy the equation

$$y(z) + \lambda \int_{-\pi}^{+\pi} \sin (k \sin z \cos \theta) y(\theta) d\theta = 0,$$

and that both satisfy the equation

$$y(z) + \lambda \int_{-\pi}^{+\pi} e^{ik \sin z \sin \theta} y(\theta) d\theta = 0.$$

These equations have solutions only when λ satisfies the equation $D(\lambda) = 0$, the equation for the characteristic numbers in the theory of integral equations.

When $k = 0$ the solution of the differential equation of period π or 2π is obtained by supposing a to be the square of an integer; these solutions being 1, $\cos z$, $\sin z$, $\cos 2z$, $\sin 2z$, $\cdots$. When $k \neq 0$ there exist solutions for proper values of a which reduce to these if $k = 0$. These are denoted by $\mathrm{ce}_0(z)$, $\mathrm{ce}_1(z)$, $\mathrm{se}_1(z)$, $\mathrm{ce}_2(z)$, $\mathrm{se}_2(z)$, etc.

These solutions could be calculated from the integral equation if λ were known. But it was shown by Whittaker that the early members of the succession of functions can be found without knowing λ. His method succeeds as far as $\mathrm{ce}_6(z)$, $\mathrm{se}_6(z)$ but cannot be carried further, since to obtain the terms of the Fourier series for $\mathrm{ce}_n(z)$, $\mathrm{se}_n(z)$, $n \geqq 7$, requires the solution of a system of linear equations which involves a greater number of unknowns

than of equations. This system might be extended to an infinite one but it is not possible to obtain the solution. Another method must be followed which is exemplified by the following determination of $\mathrm{ce}_0(z)$.

14. Let

$$y = \mathrm{ce}_0(z) = a_0 + a_2 \cos 2z + a_4 \cos 4z + \cdots$$

be a solution of the differential equation. Then differentiating and substituting there results

$$-a(a_0 + a_2 \cos 2z + a_4 \cos 4z + \cdots) = -2^2 a_2 \cos 2z$$
$$-4^2 a_4 \cos 4z - \cdots + k^2[a_0/2 + (a_0/2)\cos 2z + a_2/4 + (a_2/4)\cos 2z$$
$$+ \cdots + (a_{2r}/4)\cos(2r-2)z + (a_{2r}/2)\cos 2rz + (a_{2r}/4)\cos(2r+2)z + \cdots\ ].$$

Equate coefficients of cosines of like multiples of z and the following equations are obtained:

$$(a + k^2/2)a_0 + (k^2/4)a_2 = 0,$$
$$(k^2/2)a_0 + (a + k^2/2 - 2^2)a_2 + (k^2/4)a_4 = 0,$$
$$(k^2/4)a_2 + (a + k^2/2 - 4^2)a_4 + (k^2/4)a_6 = 0,$$
$$(k^2/4)a_4 + (a + k^2/2 - 6^2)a_6 + (k^2/4)a_8 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(k^2/4)a_{2n-2} + (a + k^2/2 - 4n^2)a_{2n} + (k^2/4)a_{2n+2} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

It is necessary to consider the determinants

$$D_1 = a + k^2/2,$$

$$D_2 = \begin{vmatrix} a + k^2/2 & k^2/4 \\ k^2/2 & a + k^2/2 - 2^2 \end{vmatrix} = (a + k^2/2)^2 - 2^2(a + k^2/2) - 2(k/2)^4,$$

$$D_3 = \begin{vmatrix} a + k^2/2 & k^2/4 & 0 \\ k^2/2 & a + k^2/2 - 2^2 & k^2/4 \\ 0 & k^2/4 & a + k^2/2 - 4^2 \end{vmatrix}$$

$$= (a + k^2/2)^3 - (2^2 + 4^2)(a + k^2/2)^2 + (2^2 4^2 - 3(k/2)^4)(a + k^2/2) + 2 \cdot 4^2 (k/2)^4,$$

$$D_4 = \begin{vmatrix} a + k^2/2 & k^2/4 & 0 & 0 \\ k^2/2 & a + k^2/2 - 2^2 & k^2/4 & 0 \\ 0 & k^2/4 & a + k^2/2 - 4^2 & k^2/4 \\ 0 & 0 & k^2/4 & a + k^2/2 - 6^2 \end{vmatrix}$$

$$= (a + k^2/2)^4 - (2^2 + 4^2 + 6^2)(a + k^2/2)^3$$
$$+ (2^2\cdot 4^2 + 2^2\cdot 6^2 + 4^2\cdot 6^2 - 4(k/2)^4)(a + k^2/2)^2$$
$$- \{2^2\cdot 4^2\cdot 6^2 - (2^2 + 2\cdot 4^2 + 3\cdot 6^2)(k/2)^4\}(a + k^2/2)$$
$$- 2\cdot 4^2\cdot 6^2(k/2)^4 + 2(k/2)^8,$$

$$D_5 = \begin{vmatrix} a + k^2/2 & k^2/4 & 0 & 0 & 0 \\ k^2/2 & a + k^2/2 - 2^2 & k^2/4 & 0 & 0 \\ 0 & k^2/4 & a + k^2/2 - 4^2 & k^2/4 & 0 \\ 0 & 0 & k^2/4 & a + k^2/2 - 6^2 & k^2/4 \\ 0 & 0 & 0 & k^2/4 & a + k^2/2 - 8^2 \end{vmatrix}$$

$$= (a + k_2/2)^5 - (2^2 + 4^2 + 6^2 + 8^2)(a + k^2/2)^4$$
$$+ \{2^2\cdot 4^2 + 2^2\cdot 6^2 + 4^2\cdot 6^2 + 2^2\cdot 8^2 + 4^2\cdot 8^2 + 6^2\cdot 8^2$$
$$- 5(k/2)^4\}(a + k^2/2)^3 - \{2^2\cdot 4^2\cdot 6^2 + 2^2\cdot 4^2\cdot 8^2 + 2^2\cdot 6^2\cdot 8^2$$
$$+ 4^2\cdot 6^2\cdot 8^2 - (2\cdot 2^2 + 3\cdot 4^2 + 3\cdot 6^2 + 4\cdot 8^2)(k/2)^4\}(a + k^2/2)^2$$
$$+ \{2^2\cdot 4^2\cdot 6^2\cdot 8^2 - (2^2\cdot 4^2 + 2^2\cdot 8^2 + 2^2\cdot 4^2\cdot 6^2 + 2^2\cdot 4^2\cdot 8^2$$
$$+ 3^2\cdot 6^2\cdot 8^2)(k/2)^4 - 5(k/2)^8\}(a + k^2/2)$$
$$+ 2\cdot 4^2\cdot 6^2\cdot 8^2(k/2)^4 - (2\cdot 4^2 + 2\cdot 8^2)(k/2)^8.$$

These determinants are connected by the relation

$$D_n = \{a + k^2/2 - (2n - 2)^2\}D_{n-1} - (k^2/4)^2 D_{n-2},$$

as may be verified by expanding D_n in terms of the last row. The quantities D_n regarded as functions of a have the characteristics of a Sturmian series. D_n is of the nth degree in a, and the coefficient of a^n is $+1$. Moreover it is evident from the recurrence relation if $D_{n-1}=0$ that D_n and D_{n-2} have opposite signs. Denote the number of variations of sign when a is put $=+\infty$ in the succession $D_nD_{n-1}\cdots D_1D_0(D_0=1)$ by $V_{+\infty}$ and when $a=-\infty$ by $V_{-\infty}$, and it is evident that $V_{-\infty}-V_{+\infty}=n$. Hence the equation $D_n=0$ has n real roots; it also follows that D_n and D_{n-1} have unlike signs just before each of the n roots of $D_n=0$ and like signs immediately after.

It will be proved in § 17 that the quantities a, a_0, a_2, $\cdots$ are power series in k^2 where a_0 may be put equal to unity. If it be assumed that a solution $\mathrm{ce}_0(z)$ exists which reduces to unity when $k=0$, it follows that $a=(k^2)$, $a_2=(k^2)$, $\cdots$, $a_{2n}=(k^2)$, where the notation (k^n) means divisible by k^n. Since $a+k^2/2=-(k^2/4)a_2$, then $a+k^2/2=(k^4)$. If now the solution of the first n equations is expressed in the form

$$a_0 : a_2 : \cdots : a_{2n}$$

$$= \begin{Vmatrix} a + k^2/2 & k^2/4 & 0 & \cdot & \cdot & \cdot & \cdot \\ k^2/2 & a + k^2/2 - 2^2 & k^2/4 & \cdot & \cdot & \cdot & \cdot \\ 0 & k^2/4 & a + k^2/2 - 4^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & k^2/4 & a + k^2/2 - (2n-2)^2 & k^2/4 \end{Vmatrix}$$

it appears that

$$\frac{a_{2n}}{a_0} = + \frac{4^n D_n}{k^{2n}} .$$

But $a_{2n}/a_0 = (k^2)$; hence $D_n = (k^{2n+2})$.

This allows the calculation of the successive terms of a as follows. Put $a + k^2/2 = ck^4 + \cdots$; substitute in D_2 and require that the result shall have the form (k^6). This gives $(ck^4 + \cdots - 4)(ck^4 + \cdots) - k^4/8 = (k^6)$, or $-4c - 1/8 = 0$, or $c = -1/32$. Next substitute in D_3 for $a + k^2/2$ the expression $-(1/32)k^4 + ck^6 + \cdots$ and require that the lowest power of k^2 which appears shall have a vanishing coefficient. This power is multiplied by $2^2 \cdot 4^2 \cdot c$. Hence $c = 0$. The process may be continued; at each stage a new coefficient is found unambiguously, and finally the expression for a in powers of k.

15. When a is known the coefficients $a_2, a_4, \cdots$ are at once determined, but it is preferable to arrange the series for $\mathrm{ce}_0(z)$ in powers of k, or to put

$$\mathrm{ce}_0(z) = 1 + b_2 k^2 + b_4 k^4 + \cdots ,$$

where $b_2, b_4, \cdots$ are finite trigonometric series. Substitute in the differential equation and give a the value just found in powers of k. There results

$$b_2'' k^2 + b_4'' k^4 + \cdots + (a + k^2/2)(1 + b_2 k^2 + b_4 k^4 + \cdots) + (k^2/2)(1 + b_2 k^2 + b_4 k^4 + \cdots) \cos 2z = 0.$$

If the coefficients of $k^2, k^4, \cdots$ are equated to zero, a system of linear differential equations with constant coefficients for $b_2, b_4, \cdots$ results:

$$b_2'' + \tfrac{1}{2} \cos 2z = 0,$$

$$b_4'' - \frac{1}{32} + \frac{1}{2} b_2 \cos 2z = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot .$$

This simultaneous system may be solved, and the complementary functions must be put equal to zero to obtain $\mathrm{ce}_0(z)$. It is found that

$$\mathrm{ce}_0(z) = 1 + (k^2/8) \cos 2z + (k^4/2^9) \cos 4z + \cdots .$$

The value of a is equal to

$$a = -k^2/2 - k^4/2^5 + 0 \cdot k^6 - \frac{7}{389 \cdot 2} k^8 + \cdots .$$

It is necessary to examine the statement made above that the values of the coefficients found for the expansion of a are determined unambiguously. The development of a has been obtained by substitution of the series

$$c_1 k^2 + \cdots + c_n k^{2n} + c_{n+1} k^{2n+2} + \cdots$$

for a, and by determination of $c_1, \cdots, c_n$ so that $D_n = (k^{2n+2})$. For example if for a is substituted

$$-k^2/2 - (1/32)k^4 + 0 \cdot k^6 + c_4 k^8 + \cdots$$

it is found that

$$D_1 = (k^4), D_2 = (k^6), D_3 = (k^8);$$

but

$$D_4 = (a + k^2/2 - 6^2)D_3 - (k^2/2)^2 D_2 = (k^8),$$

and these equations are true whatever c_4. If c_4 can be determined to make $D_3 = (k^{10})$, it follows that $D_4 = (k^{10})$, since $(k^2/4)^2 D_2 = (k^{10})$. When $k^2 = 0$, the roots of $D_n = 0$ are distinct. Then from the principles of algebraic functions, there exists a unique series $c_1 k^2 + \cdots$ which expresses that root of the equation $D_n = 0$ which vanishes with k^2. The coefficients of this series may be determined seriatim by the conditions that $D_n = (k^2)$, $D_n = (k^4)$, etc. Applying this theorem to D_4 it follows that the coefficients of the first three terms are $-1/2$, $-1/32$, 0. Hence it must be possible to determine c_4 so as to make $D_4 = (k^{10})$. The subsequent terms in the expansion of a may be similarly determined.

The quantities a_{2n}, D_n are really divisible by higher powers of k than has yet appeared; in fact $a_{2n} = (k^{2n})$ and $D_n = (k^{4n})$. This may be proved from the system of equations for b_2, b_4, etc., which are

$$b_2'' + c_n b_0 + c_{n-1} b_2 + c_{n-2} b_4 + \cdots + c_2 b_{2n-4} + \tfrac{1}{2}(\cos 2z) b_{2n-2} = 0 \quad (n = 1, \cdots)$$

where $a + k^2/2 = c_2 k^4 + \cdots$, $c_1 = 0$, $b_0 = 1$. It is evident that b_{2n} is the first unknown which involves $\cos 2nz$, so that k^{2n} is the lowest power of k which occurs in the coefficient of $\cos 2nz$; hence $a_{2n} = (k^{2n})$. Since

$$\frac{a_{2n}}{a_0} = \pm \frac{4^n D_n}{k^{2n}},$$

it follows that $D_{2n} = (k^{4n})$.

16. The method of determination of $\mathrm{ce}_0(z)$ may be applied to the calculation of all the functions $\mathrm{ce}_{2n}(z)$. Consider the next even function of period π, and write

$$\mathrm{ce}_2(z) = a_0 + a_2 \cos 2z + \cdots .$$

In the system of equations for $a_0, a_2, \cdots$ (§ 14), put $a_2 = 1$. When $k = 0$, a reduces to 2^2 and a_0, a_4, a_6 vanish, and so are of the form (k^2). Upon reference to the solution given for $a_0, \cdots, a_{2n}$, it is evident that $D_n = (k^{2n})$, $n \geqq 2$. This allows the calculation of a by a method analogous to that used in the case of $\mathrm{ce}_0(z)$. From the first equation, $a_0 = (k^2)$. Substitute in the second and it follows that $a + k^2/2 - 2^2 = (k^4)$. Put

$$a = 2^2 - k^2/2 + ck^4 + \cdots ,$$

and since $D_2 = (k^4)$ whatever c, make $D_3 = (k^6)$. Then

$$c = +\frac{5}{3 \cdot 2^6} .$$

Put

$$a = 2^2 - \frac{k^2}{2} + \frac{5}{3 \cdot 2^6} k^4 + ck^6 + \cdots ,$$

and make $D_4 = (k^8)$. Then $c = 0$, etc. Hence

$$a = 4 - \frac{k^2}{2} + \frac{5}{3 \cdot 2^6} k^4 + 0 \cdot k^6 + \cdots .$$

It is needless to repeat the arguments which justify this procedure. The function $\mathrm{ce}_2(z)$ is determined by a system of differential equations as was $\mathrm{ce}_0(z)$; the result is

$$\mathrm{ce}_2(z) = \cos 2z + \frac{k^2}{2^5}(\cos 4z - 2) + \frac{k^4}{6 \cdot 2^{10}} \cos 6z + \cdots .$$

The results of calculation are here illustrated; $a^{(n)}$ denotes the value of a for $\mathrm{ce}_n(z)$ and $a_{(n)}$ the value of a for the nth odd function $\mathrm{se}_n(z)$:

$$a^{(0)} = -k^2/2 - k^4/2^5 + 0 \cdot k^6 + \cdots ,$$

$$a^{(1)} = 1 - \frac{3}{4} k^2 - \frac{1}{2^7} k^4 + \frac{1}{3 \cdot 2^{12}} k^6 + \cdots ,$$

$$a^{(2)} = 4 - \frac{1}{2} k^2 + \frac{5}{3 \cdot 2^6} k^4 + 0 \cdot k^6 + \cdots ,$$

$$a^{(3)} = 9 - \frac{1}{2}k^2 - \frac{1}{2^8}k^4 - \frac{1}{2^{12}}k^6 + \cdots,$$

$$a^{(4)} = 16 - \frac{1}{2}k^2 - \frac{1}{2^5 \cdot 15}k^4 - \frac{11}{2^9 \cdot 45}k^6 + \cdots,$$

$$a_{(1)} = 1 - k^2/2^2 - k^4/2^7 + \cdots,$$

$$\mathrm{ce}_0(z) = 1 + (k^2/8)\cos 2z + (k^4/2^9)\cos 4z + \cdots,$$

$$\mathrm{ce}_1(z) = \cos z + (k^2/2^5)\cos 3z + (k^4/2^{10})(\cos 5z - (1/3)\cos 3z) + \cdots,$$

$$\mathrm{ce}_2(z) = \cos 2z + (k^2/2^5)((2/3)\cos 4z - 2) + \frac{k^4}{6 \cdot 2^{10}}\cos 6z + \cdots,$$

$$\mathrm{ce}_3(z) = \cos 3z + (k^2/2^5)(-\cos z + \tfrac{1}{2}\cos 5z) + (k^4/2^{10})(\cos z + (1/10)\cos 7z) + \cdots,$$

$$\mathrm{ce}_4(z) = \cos 4z + (k^2/2^5)((2/5)\cos 6z - (2/3)\cos 2z) + \frac{k^4}{2^{10}}\left(\frac{1}{15}\cos 8z + \frac{88}{45}\cos 2z + \frac{1}{3}\right) + \cdots,$$

. . . . . .

$$\mathrm{se}_1(z) = \sin z + (k^2/2^5)\sin 3z + (k^4/2^{10})((1/3)\sin 5z + \sin 3z) + \cdots;^*$$

. . . . . .

The differential equation

$$\frac{d^2y}{dz^2} + (a + k^2\cos^2 z)y = 0$$

can be transformed into itself. Put $z = z' + \pi/2$; then

$$\frac{d^2y}{dz'^2} + (a + k^2/2 - (k^2/2)\cos^2 z')y = 0.$$

This transformation makes possible the calculation of a second function and a second value of a, when a first function and a first value of a are given. For example if the transformation is applied to $\mathrm{ce}_1(z)$ there results the function $-\mathrm{se}_1(z)$, and $a^{(1)}$ is transformed into $a_{(1)}$, etc.; $\mathrm{ce}_0(z)$ and a_0 are exceptional. It is evident that on replacing z by $z' + \pi/2$ and k^2 by $-k^2$, $\mathrm{ce}_0(z)$ is transformed into itself. Hence a_0 is transformed into itself. But this proves that all powers of k in a_0 after the first term are of the form $4n$.

* These calculations have been carried further by Dr. Saund, in his doctor dissertation at the University of California.

17. The assumption has been made, in what precedes, that periodic solutions exist, and that the quantities $a, \cdots, a_n, \cdots$ are analytic functions of k. It remains to justify these assumptions. In what follows two proofs of the existence of periodic solutions will be given. It may be seen that periodic solutions exist for every value of k^2. For consider the equations $a_{11}+a_{22}=\pm 2$ (§ 7) in which $\lambda=-k^2$, and $h^2=-a$. For a given value, $\lambda=\lambda_0$, $a_{11}+a_{22}$ is an integral function of h. In accordance with the general theory of Picard,* if one of these equations has not an infinite number of solutions the other one has an infinite number. If the equation $a_n+a_{22}=2$ has an infinite number of solutions there exists for the value $\lambda=\lambda_0$ an infinite number of functions having period π. If $a_{11}+a_{22}=-2$ has an infinite number of solutions, there exists an infinite number of functions $f(x)$ such that $f(x+\pi)=-f(x)$, or $f(x+2\pi)=f(x)$. Hence there exists for all values of λ an infinite number of functions having the period π or 2π. This theorem is not sufficient to prove the existence, for all values of k^2, of the functions $\mathrm{ce}_n(z)$, $\mathrm{se}_n(z)$ for which formal developments have just been obtained.

A determination of all periodic solutions of the differential equation will now be given in the form of a proof that the series for $\mathrm{ce}_n(z)$, $\mathrm{se}_n(z)$ and $a^{(n)}$, $a_{(n)}$ converge for all values of k^2. Write the system of equations for $a_0, a_2, \cdots$ as follows. Put

$$\frac{k^2}{4(a+k^2/2-(2n)^2)}=x_n,$$

and $a+k^2/2=v$; then the equations beginning at the third are

$$\begin{aligned} x_2c_1+c_2+x_2c_3&=0,\\ x_3c_2+c_3+x_3c_4&=0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\end{aligned}$$

where $c_n=a_{2n}$. This is a system of the same form as that previously treated (§ 4). A solution such that $c_n\to 0$ was found to be given by

$$c_1:c_2:\cdots c_n:\cdots=(x_2):-x_2(x_3):\cdots:(-1)^n x_2\cdots x_{n+1}(x_{n+2}):\cdots.$$

It is supposed that $v\neq(2n)^2$ so that $x_n\neq\infty$. Substitute the values of c_1, c_2 in the first and second equations (§ 14) and they become

$$va_0+(k^2/4)\begin{vmatrix} 1 & x_2 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}=0,$$

* Picard, *Traité d'Analyse*, vol. 3, p. 346.

$$(k^2/2)a_0 + (v - 2^2)\begin{vmatrix} 1 & x_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} - x_2(k^2/4)\begin{vmatrix} 1 & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Eliminating a_0 there results

$$v(v - 2^2)\begin{vmatrix} 1 & x_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} - x_2 v(k^2/4)\begin{vmatrix} 1 & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} - (k^4/8)\begin{vmatrix} 1 & x_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

This may be written in the form

$$d = \begin{vmatrix} a + k^2/2 & k^2/4 & 0 & 0 & 0 & \cdot & \cdot \\ k^2/2 & a + k^2/2 - 2^2 & k^2/4 & 0 & 0 & \cdot & \cdot \\ 0 & x_2 & 1 & x_2 & 0 & \cdot & \cdot \\ 0 & 0 & x_3 & 1 & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

If $D_n = (a+k^2/2-4^2)\,(a+k^2/2-6^2)\cdots(a+k^2/2-(2n-2)^2)d_n$, then $d_n \to d$ as $n \to \infty$. This follows since

$$d_n = \begin{vmatrix} a + k^2/2 & k^2/4 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ k^2/2 & a + k^2/2 - 2^2 & k^2/4 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & x_2 & 1 & x_2 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & x_3 & 1 & x_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & x_{n-1} & 1 \end{vmatrix}.$$

This determinant is convergent as may be seen on applying the test for convergence used in § 2. Hence $d_n \to d$, $n \to \infty$. The equation for v given above may evidently be written $v = P_1(v, k^2)$ where every term in $P_1(v, k^2)$ is divisible by either v^2 or k^2; hence $v = P(k^2)$ and it is clear that $P(k^2) = (k^4)$. The value of a corresponding to v is therefore $-k^2/2 + ck^4 + \cdots$, the terms of which have been calculated above (§ 14). It will be seen that the equation $d = 0$ has an infinite number of real roots of which this value of a is the first, and they reduce when $k = 0$ to $0, 2^2, 4^2, \cdots$. This is proved in the following way. If the determinant d is expanded in terms of the row containing x_n the equation $d = 0$ may be written in the form

$$v - (2n)^2 = P_1(v - (2n)^2, \quad k^2),$$

where every term in $P_1(v-(2n)^2, k^2)$ is divisible by $(v-(2n)^2)$ or k^2. Accordingly $v-(2n)^2=P(k^2)$, or

$$a = (2n)^2 - k^2/2 + P(k^2).$$

These values of a may all be calculated from the equations $D_n=0$ in the manner carried out in § 14.

In this way the roots which tend to $(2n)^2$ as $k\to 0$ are found. It may be proved that there are no others, or more precisely, no root of the equation $d=0$ tends to a limit, $l\neq(2n)^2$, and no root increases indefinitely as $k\to 0$. The first statement is true since a and k^2 satisfy the equation $a_{11}+a_{22}-2=0$. If a root of this equation in a is bounded as $k\to 0$ it must have a limit. This limit is a root of the equation $a_{11}+a_{22}-2=0$ in which k has been put equal to zero. Further a root of the equation $d=0$ cannot $\to\infty$ when $k\to 0$. If $v\to\infty$ then $|v|$ takes the value $(2p+1)^2$ infinitely often, and

$$|v-(2n)^2| \geqq ||v|-(2n)^2| = |(2p+1)^2-(2n)^2|.$$

The least value of this quantity for a given p is furnished by $n=p$, so that $|v-(2n)^2|\geqq 4p+1$; hence $x_n\to 0$ for these values of $|v|$. But the equation $d=0$ may be written

$$v(v-2)^2\begin{vmatrix} 1 & x_0 & 0 & 0 & \cdot & \cdot & \cdot \\ 2x_1 & 1 & x_1 & 0 & \cdot & \cdot & \cdot \\ 0 & x_2 & 1 & x_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

and cannot be satisfied if $|v|=2p+1$, $p\to\infty$. Hence as $k\to 0$ the roots of $d=0\to 0, 2^2, 4^2, \cdots$. The same reasoning shows that when $k\to k_0\neq 0$ no root of the equation $d=0\to\infty$.

If a value of a satisfying $d=0$ is obtained the values of the coefficients $a_2, a_4, \cdots$ are given in two different forms in what precedes, viz.

$$a_{2n} = (-1)^n\frac{4^nD_n}{k^{2n}} \qquad (n=0,1,\cdots, D_0=1),$$

and $a_{2n}=c_n/c_0$ where

$$c_n = (-1)^n x_2, \cdots, x_{n+1}(x_{n+2}) \qquad (n=1,2,\cdots),$$

$$c_0 = \frac{k^2}{4(a+k^2/2)}a_2 = -x_0(x_2).$$

The series $1+a_2\cos 2z+\cdots$ is convergent because the ratio

$$r_n = \frac{a_{2n}}{a_{2n-2}} = -\frac{x_2 \cdots x_{n+1}(x_{n+2})}{x_2 \cdots x_n(x_{n+1})} = -\frac{x_{n+1}(x_{n+2})}{(x_{n+1})}, \quad n > 1;$$

and since $(x_n) \to 1$ as $n \to \infty$ it follows that

$$r_n = -\frac{k^2}{4\{a + k^2/2 - (2n+2)^2\}} + \epsilon.$$

Hence the series $1 + a_2 \cos 2z + \cdots$ converges for all values of z.

By a similar reasoning the existence of the odd functions as well as those of period 2π may be proved.

It has been assumed in the preceding analysis that $a + k^2/2 \neq (2n)^2$. If $a + k^2/2 = (2n)^2$ the proof of convergence may be effected by beginning at the nth equation for $a_0, a_2, \cdots$ and expressing the solution of the system

$$\begin{aligned} x_{n+1}c_n + c_{n+1} + x_{n+1}c_{n+2} &= 0, \\ x_{n+2}c_{n+1} + c_{n+2} + x_{n+2}c_{n+3} &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \end{aligned}$$

in the same manner as before.

The functions denoted $\mathrm{ce}_n(z)$, $\mathrm{se}_n(z)$ are obtained from the expressions just given for the periodic solution by dividing by the coefficient of $\cos nz$ or of $\sin nz$, so that these terms have coefficients equal to unity. This division is possible if k is sufficiently small since the divisors are $\neq 0$.

Finally the existence of periodic solutions has been proved and expressions for all of them for all values of k^2 have been determined.

18. The series obtained for the quantities a and the periodic functions converge if k is sufficiently small, but the radius of convergence has not been found. Consider the values of a which belong to $\mathrm{ce}_{2n}(z)$; it may be proved that the singular points of a are those values of k for which two or more values of a coincide. For if k_0 is the singular value of k then when $k \to k_0$, it was proved above that $a \to 1$, a finite value; and if the values of a corresponding to $k = k_0$ are distinct, each of the values of a is by the theory of implicit functions a series of integral powers of $k - k_0$. But the value k_0 would not be singular if the values of a are so expressed; hence the values of a are not distinct if $k = k_0$.

If $f_n(z), f_m(z)$ are two periodic functions corresponding to two values of a which become equal, then $f_n(z) = \rho f_m(z)$, ρ independent of z, when $k = k_0$. Since by the theory of integral equations

$$\int_{-\pi}^{\pi} f_n(z) f_m(z) dz = 0,$$

it follows that if $k=k_0$,

$$\int_{-\pi}^{\pi} f_n^2(z)dz = 0.$$

The value of this integral may be expressed in terms of $a_2, a_4, \cdots$ giving

$$\int_{-\pi}^{\pi} f_n^2(z)dz = (1 + \tfrac{1}{2}a_2^2 + \tfrac{1}{2}a_4^2 + \cdots)2\pi.$$

Hence if for $k=k_0$ two values of a become equal, and if in

$$a_{2n} = (-1)^n \frac{4^n D_n}{k^{2n}},$$

the value with which they coincide is substituted, then

$$1 + \tfrac{1}{2}a_2^2 + \tfrac{1}{2}a_4^2 + \cdots = 0.$$

It follows that the singular values k_0^2 are not real. To prove this it is necessary to prove first that if k^2 is real the values of a are real. Suppose that for a real value of k^2, a has the value $a'+ia''$ and y has the value y_1+iy_2. Then

$$y_1'' + iy_2'' + (a' + ia'' + k_2\cos^2 z)(y_1 + iy_2) = 0,$$

or

$$y_1'' + (a' + k^2\cos^2 z)y_1 - a''y_2 = 0,$$

$$y_2'' + (a' + k^2\cos^2 z)y_2 + a'y_1 = 0.$$

It follows that

$$\frac{d}{dz}(y_2y_1' - y_1y_2') = a''(y_1^2 + y_2^2),$$

and integrating from 0 to π that

$$0 = a'' \int_0^{\pi} (y_1^2 + y_2^2)dz$$

since y_1 and y_2 have the period π. Hence $a''=0$ or a is real. If a and k^2 are real, a_{2n} is real so that

$$1 + \tfrac{1}{2}a_2^2 + \tfrac{1}{2}a_4^2 + \cdots \neq 0.$$

Hence k_0^2 is not real if k_0 is a singular value. Since the singular values of k correspond to equal values of a the equation $d=0$ cannot have a multiple root if k^2 is real.

A numerical estimate of the values of k^2 and a which are singular may be found from the polynomials D_n by finding the values of k^2 for which

the equation $D_n=0$ has coincident roots. It appears that approximately $k^2=5.8\,i$, $a=2.1-2.9i$ for the singular point nearest zero.

19. The equation of the elliptic cylinder cannot have two periodic solutions of period π or πi for the same values of a and k^2. If $\mu=0$ two solutions are given by the formulas

$$f(z) = \frac{1}{\phi(0)} \sum_{-\infty}^{+\infty} \phi(n) J_n(\lambda e^{-z}) J_n(\lambda e^{z}),$$

$$\psi(z) = \frac{1}{\phi(0)} \sum_{-\infty}^{+\infty} \phi(n) J_n(\lambda e^{-z}) K_n(\lambda e^{z}).$$

The second solution is distinct from the first, since

$$K_n(x) = c\,(\log x) J_n(x) + P_n(x),$$

where c is not zero and $P_n(x)$ is a power series in x. This solution does not vanish identically since this would require $\phi(n)=0$ for every n. Since every solution is a linear combination of $f(z)$ and $\psi(z)$, and since every solution is periodic if two independent solutions are periodic, $\psi(z)$ must be periodic. But

$$\psi(z) = c\,(\log \lambda) e^{z} f(z) + \sum_{-\infty}^{+\infty} \phi(n) J_n(\lambda e^{-z}) P_n(\lambda e^{z}),$$

which is not periodic. Hence the equation of the elliptic cylinder never has two independent periodic solutions.

An incidental result of the theorem that two values $a_{(2n)}$, $a_{(2m)}$ cannot coincide when k^2 is real is that $a_{(2m)} > a_{(2n)}$ if $m>n$. For if $a_{(2m)} < a_{(2n)}$ for any value of k, letting k^2 diminish to zero, there must for some value of k^2 be found the equality $a_{(2m)} = a_{(2n)}$, since when $k^2=0$, $a_{(2m)} = 2m^2$ and $a_{(2n)} = (2n)^2$. But $a_{(2m)} \neq a_{(2n)}$ for any real value of k^2. Hence $a_{(2m)} > a_{(2n)}$ if $m>n$.

It may further be proved that if k^2 is real the value of da/dk^2 is negative. First it may be proved that $da/dk^2 \neq 0$ for any real value of k^2. For suppose $da/dk^2=0$ and let $w=dy_1/dk^2$ where y_1 is a periodic solution. Then if y_2 is an independent solution it may be supposed that $y_2 = zy_1 + y_3$ where y_3 is periodic. The function w must satisfy the equation

$$\frac{d^2w}{dz^2} + (a + k^2 \cos z) w = -\, y_1 \cos^2 z,$$

and the solution of this equation is known to be

$$w = Ay_2 + \frac{y_2}{c}\int -(\cos^2 z)y_1^2 dz + By_1 - \frac{y_1}{c}\int -(\cos^2 z)y_1 y_2 dz,$$

where

$$c = y_1\frac{dy_2}{dz} - y_2\frac{dy_1}{dz} \neq 0.$$

If

$$-(\cos^2 z)y_1^2 = A_0 + A_1\cos 2z + \cdots$$

it is found that

$$w = \tfrac{1}{2}A_0 z^2 y_1 + zF(z) + G(z),$$

where $F(z)$ and $G(z)$ are periodic functions of z. Hence if w is periodic, $A_0 = 0$. But

$$A_0 = -\frac{1}{2\pi}\int_0^{2\pi}(\cos^2 z)y_1^2 dz,$$

hence

$$\int_0^{2\pi}(\cos^2 z)y_1^2 dz = 0$$

which is impossible if k^2 is real. Hence $da/dk^2 \neq 0$. The values of da/dk^2 when $k=0$ are known however from the series for a, viz., $-k^2/2 - k^4/32$, $1 - 3k^2/4 - \cdots$, and are seen to be $-1/2$, $-1/3$, $-1/2$, $\cdots$, the values being $-1/2$ after the second. Hence $da/dk^2 < 0$ for all real values of k^2, which was to be proved.

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ON A CLASS OF INTEGRAL EQUATIONS WITH DISCONTINUOUS KERNELS*

BY

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CHAPTER 1. INTRODUCTION

This paper is given to the discussion of the integral equation in which the kernel $K(x, \xi)$ is continuous, but possesses first partial derivatives which are discontinuous along the line $\xi = x$. It is closely related to a previous paper† in which the case of a discontinuous kernel has been treated. The motivation of the problem and the origin of the work is there discussed. Suffice it to remark here that the present paper originated in a thesis prepared under Birkhoff and presented at Radcliffe College in 1921.‡

The integral equation with a kernel having the properties of discontinuity mentioned above is also the subject of a recent paper by A. Hammerstein.§ He considers, however, only the case in which the kernel is also symmetric. Assuming the existence of infinitely many positive characteristic values he obtains an asymptotic form for the characteristic functions.

The present paper, without the hypothesis of symmetry, establishes the existence and derives the asymptotic forms of the characteristic values, and gives for the characteristic functions an asymptotic expression which is considerably more explicit than that obtained by Hammerstein for a more restricted case. The closure of the set of solutions is discussed and the problem of the expansion of an arbitrary function in a series of solutions is treated. The methods used are on the whole the same as those used in the earlier paper referred to above, and where the procedure is similar it has been passed over briefly here. One improvement, however, which is also applicable to the case there treated may be mentioned, namely, the proof that the set of characteristic functions is closed has been materially simplified. This was accomplished by the use of an extension of Birkhoff's theorem on

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† Langer, these Transactions, vol. 28 (1926). This paper will be referred to in the text simply as paper 1.

‡ By Eleanor Pairman, now Mrs. Eleanor P. Brown.

§ *Über die asymptotische Darstellung der Eigenfunktionen linearer Integralgleichungen*, Mathematische Annalen, 1924, p. 113. Cf. also his paper in Mathematische Annalen, 1926, p. 102, where an analogous situation for the integral equation in two variables is treated.

closure, and of a general theorem on the closure of the solutions of an integral equation.*

The final chapter outlines the theory of an equation which does not satisfy the general hypotheses made, and shows that under such circumstances the solutions may differ radically in character from those of the cases more fully treated.

CHAPTER 2. THE NORMALIZATION OF THE EQUATION

1. **The type of equation treated.** The integral equation to be discussed is of the form

$$y(t) + \lambda^2 \int_\alpha^\beta \Gamma(t,\tau) y(\tau) d\tau = 0, \tag{1}$$

where λ is a complex parameter, while the kernel $\Gamma(t, \tau)$ is real and satisfies the following hypotheses:

(i) $\Gamma(t, \tau)$ is continuous in the region

$$\overline{R}\begin{cases} \alpha \leqq t \leqq \beta \\ \alpha \leqq \tau \leqq \beta, \end{cases}$$

and possesses partial derivatives of order 2 which are continuous in the open regions

$$\overline{R}'\begin{cases} \alpha \leqq \tau < t \\ \alpha \leqq t \leqq \beta, \end{cases} \qquad \overline{R}''\begin{cases} t < \tau \leqq \beta \\ \alpha \leqq t \leqq \beta, \end{cases}$$

and approach finite limiting values on the boundary $\tau = t$.

(ii) $$\Gamma_t(t,\tau)\Big]_{\tau=t-}^{\tau=t+} = \phi(t) \not\equiv 0.$$

(iii) The functions

$$\frac{\partial^2 \Gamma(t,\tau)}{\partial t^{2-j} \partial \tau^j}\Big]_{\tau=t-}^{\tau=t+} \qquad (j = 0,1,2)$$

each possess a continuous derivative on the interval (α, β).

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* Langer, *Three theorems on closure of biorthogonal systems of functions*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 97–105. We note here that in this paper the requirement that the solutions of the equation numbered 8 form a biorthogonal set should have appeared as part of the hypothesis.

2. **The change of variables.** Under the hypotheses made the function $\phi(t)$ possesses a continuous derivative of second order.* There is no loss of generality moreover in assuming $\phi(t)>0$. The quantities

$$x = \int_a^t \{\phi(t)\}^{1/2} dt, \qquad \xi = \int_a^\tau \{\phi(\tau)\}^{1/2} d\tau \tag{2}$$

may serve, therefore, as new independent variables. In terms of them equation (1) takes the form

$$\bar{y}(x) = \lambda^2 \int_a^b \Omega(x,\xi)\bar{y}(\xi)d\xi, \tag{3}$$

where

$$\bar{y}(x) \equiv y(t), \qquad a = 0, \qquad b = \int_a^\beta \{\phi(\tau)\}^{1/2} d\tau,$$

and

$$\Omega(x,\xi) \equiv -\frac{\Gamma(t,\tau)}{\{\phi(\tau)\}^{1/2}}.$$

The kernel $\Omega(x, \xi)$ of this equation is continuous, while

$$\Omega_x(x,\xi)\Big]_{\xi=x-}^{\xi=x+} \equiv -1.$$

Let the function $\psi(x)$ be defined now by the relation

$$\Omega_{xx}(x,\xi)\Big]_{\xi=x-}^{\xi=x+} = \psi(x).$$

Then with the introduction of

$$u(x) = \bar{y}(x)e^{(1/2)\int_a^x \psi(x)dx} \tag{4}$$

as a new dependent variable the equation (3) in turn takes the form

$$u(x) = \lambda^2 \int_a^b Q(x,\xi)u(\xi)d\xi, \tag{5}$$

where

$$Q(x,\xi) = \Omega(x,\xi)e^{(1/2)\int_\xi^x \psi(x)dx}.$$

It is to be observed that

$$Q_{xx}(x,\xi)\Big]_{\xi=x-}^{\xi=x+} \equiv 0.$$

* Cf., e.g., paper 1, §2.

3. **The equation of normal form.*** The kernel $Q(x, \xi)$ may possibly be orthogonal on the interval (a, b) to certain continuous functions either of x or of ξ, and if this is the case either equation (5) or its associated equation will fail to satisfy certain hypotheses which must subsequently be made. We shall make, then, the following hypothesis concerning the given equation, namely:

(iv) The kernel $Q(x, \xi)$ is orthogonal to at most a finite number r of linearly independent functions $\psi_i(x)$ or to at most a finite number s of linearly independent functions $\phi_i(\xi)$, these functions being possessed of continuous derivatives of order 2. If $r \neq 0$, $s \neq 0$, then with proper assignment of subscripts the system of functions

$$\{\phi_i(x), \psi_i(x)\} \qquad (i = 1, 2, \cdots, p)$$

may be made biorthogonal and normal,† where p is the lesser of the two integers r and s, or $p = r = s$.

Under this hypothesis we may suppose that

$$\int_a^b \phi_i(x)\psi_j(x)dx = \delta_{ij} \qquad (i, j = 1, 2, \cdots, p), \tag{6}$$

where δ_{ij} is zero or one according as $i \neq j$ or $i = j$.

Consider now the integral equation

$$u(x) = \lambda^2 \int_a^b K(x, \xi)u(\xi)d\xi, \tag{7}$$

where

$$K(x, \xi) = Q(x, \xi) + \sum_{i=1}^{p} \phi_i(x)\psi_i(\xi).$$

If we assume the kernel of this equation to be orthogonal on (a, b) to a continuous function $\sigma(x)$, it follows that

$$\int_a^b Q(x, \xi)\sigma(x)dx + \sum_{i=1}^{p} \psi_i(\xi) \int_a^b \phi_i(x)\sigma(x)dx = 0.$$

Multiplying this equation by $\phi_k(x)$ and integrating we find because of (6) that

$$\int_a^b \phi_k(x)\sigma(x)dx = 0 \qquad (k = 1, 2, \cdots, p),$$

* This step in the normalization was not made in paper 1, but is equally applicable to the problem there treated.

† Cf. Goursat, *Cours d'Analyse*, vol. 3, p. 392.

and hence because of the preceding equation that $\sigma(x)$ must be orthogonal also to $Q(x, \xi)$. Accordingly $\sigma(x)$ must be a linear combination of the functions $\psi_i(x)$ and since it is orthogonal to $\phi_k(x)$ it follows that if $p=r$ then $\sigma \equiv 0$. Similarly it can be shown that if $K(x, \xi)$ is orthogonal to a continuous function $\omega(\xi)$, then $\omega \equiv 0$ if $p=s$. It will be recalled that either $p=r$, or $p=s$, or both.

It is clear that the features of discontinuity of the kernel $Q(x, \xi)$ are preserved in the kernel $K(x, \xi)$. If the definitions of the partial derivatives of $K(x, \xi)$ are completed by assigning to them on the line $\xi=x$ their limiting values as $\xi \to x, \xi < x$ the equation (7) will be said to represent the normal form. Its distinguishing characteristics are the following:

(A) There exists either no function of x or no function of ξ which is continuous with its second derivative, is orthogonal to $K(x, \xi)$, and is not identically zero.

(B) $K(x, \xi)$ is continuous in the region

$$R\begin{cases} a \leqq x \leqq b \\ a \leqq \xi \leqq b, \end{cases}$$

and possesses partial derivatives to those of order $n \geqq 2$ which are continuous in the regions

$$R_1\begin{cases} a \leqq \xi \leqq x \\ a \leqq x \leqq b, \end{cases} \quad \text{and} \quad R_2\begin{cases} x < \xi \leqq b \\ a \leqq x \leqq b, \end{cases}$$

and in the latter region approach finite limiting values at each point of the boundary $\xi=x$.

(C) $$K_x(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv -1, \qquad K_\xi(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv 1.$$

(D) $$K_{xx}(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv K_{x\xi}(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv K_{\xi\xi}(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv 0.$$

4. **The relation between the given equation and the normal equation.** Inasmuch as the functions $Q(x, \xi)$ and $\sum_{i=1}^{p}\phi_i(x)\psi_i(\xi)$ are orthogonal, the sets of characteristic values and functions of (7) are composed respectively of the corresponding sets for the two equations, namely (5), and the equation with kernel $\sum_{i=1}^{p}\phi_i(x)\psi_i(\xi)$.*

* Goursat, *Cours d'Analyse*, vol. 3, p. 402.

This latter equation is found, however, to admit of the single characteristic value $\lambda=1$, with $\phi_1(x), \cdots, \phi_p(x)$ constituting a complete set of characteristic functions.

THEOREM 1. *Every integral equation of type* (1) *whose kernel possesses the properties* (i), (ii), (iii), (iv) *may be associated with an equation of normal form as defined above.*

CHAPTER 3. TRANSFORMATION OF THE INTEGRAL EQUATION INTO AN INTEGRO-DIFFERENTIAL SYSTEM

5. **The auxiliary differential system.** We shall make the hypothesis that for the normal integral equation in hand the number n under property (B), §3, satisfies the relation

$$n \geqq 4. \tag{v}$$

Then we consider in conjunction with this equation the differential system

$$\begin{aligned} &y''(x) = 0, \\ &L_i(y) = \mu_{i1}y(a) + \mu_{i2}y'(a) + \nu_{i1}y(b) + \nu_{i2}y'(b) = 0 \qquad (i = 1,2), \end{aligned} \tag{8}$$

and its adjoint system

$$\begin{aligned} &z''(x) = 0, \\ &M_i(z) = \bar{\mu}_{i1}z(a) + \bar{\mu}_{i2}z'(a) + \bar{\nu}_{i1}z(b) + \bar{\nu}_{i2}z'(b) = 0. \end{aligned}$$

The coefficients μ_{ij}, ν_{ij} will be regarded as parameters to be chosen arbitrarily subject to the restriction that they shall not satisfy certain relations (finite in number) which arise in the deductions. Thus to begin with

(9) μ_{ij}, ν_{ij} shall be chosen so that system (8) is incompatible.

This restriction insures the existence of the Green's function $G(x, \xi)$. If we designate by $L_i(\omega)$, when $\omega=\omega(x, \xi)$ is any function of two variables, the the result produced by the operation L_i on ω considered as a function of x, and by $M_i(\omega)$ the result produced by M_i on ω as a function of ξ, then with proper definitions for the partial derivatives of $G(x, \xi)$ on the line $\xi=x$ we have

I. $G(x, \xi)$ is continuous in R and its partial derivatives are continuous in R_1 and R_2.

II. $$G_x(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv -1, \qquad G_\xi(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv 1.$$

III. $$G_{xx}(x,\xi) \equiv G_{\xi\xi}(x,\xi) \equiv 0, \qquad G_{x\xi}(x,\xi)\Big]_{\xi=x}^{\xi=x+} \equiv 0.$$

IV. $$L_i(G) = M_i(G) \equiv 0.$$

6. **The function $K(x, \xi)$.** We abbreviate by setting

$$W_i(x) = M_i(K), \qquad V_i(\xi) = L_i(K) \qquad (i = 1,2). \tag{10}$$

These functions are continuous together with their first $(n-1)$ derivatives on the interval (a, b). Moreover

$$L_i(W_j) = M_j(V_i), \tag{11}$$

for it is found by direct substitution and the use of properties (C) and (D) of the kernel $K(x, \xi)$ that (11) is equivalent to

$$\mu_{j1}\bar{\mu}_{i2} - \mu_{j2}\bar{\mu}_{i1} - \nu_{j1}\bar{\nu}_{i2} + \nu_{j2}\bar{\nu}_{i1} = 0 \qquad (i,j = 1,2), \tag{11a}$$

while the validity of this relation may be established as follows. The operators M_j and L_i are connected by the relation*

$$\int_a^b [zy'' - yz'']dx = \sum_{i=1}^{4} L_i(y)M_{5-i}(z), \tag{12}$$

where $L_i(y)$ for $i=3, 4$ may be considered as given by formulas (8) with any determination of coefficients μ_{ij}, ν_{ij}, $i=3, 4$, $j=1, 2$, for which the determinant

$$\theta = |\mu_{i1}, \mu_{i2}, \nu_{i1}, \nu_{i2}| \qquad (i = 1,2,3,4)$$

does not vanish. By direct integration of the left member of (12) its value is found to be

$$-y'(a)z(a) + y(a)z'(a) + y(b)z(b) - y(b)z'(b). \tag{12a}$$

Solving equations

$$L_i = \mu_{i1}y(a) + \mu_{i2}y'(a) + \nu_{i1}y(b) + \nu_{i2}y'(b) \qquad (i = 1, 2, 3, 4),$$

we obtain the expressions

$$y(a) = y_1, \quad y'(a) = y_2, \quad y(b) = y_3, \quad y'(b) = y_4,$$

where

$$y_i = \frac{1}{\theta}\sum_{j=1}^{4} C_{ji}L_j(y),$$

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* Cf. Birkhoff, *Boundary value and expansion problems*, these Transactions, vol. 9 (1908), p. 375.

the constant C_{ji} being the cofactor of the element in the jth row and ith column of θ. If these values are substituted in (12a), we have both members of (12) expressed as linear forms in $L_i(y)$. Equating the coefficients of $L_i(y)$ for $i=4, 3$ we obtain the forms $M_j(z), j=1, 2$. As a result we find that we may cho ose

$$\bar{\mu}_{i1} = C_{5-i,2}, \quad \bar{\mu}_{i2} = -C_{5-i,1}, \quad \bar{\nu}_{i1} = -C_{5-i,4}, \quad \bar{\nu}_{i2} = C_{5-i,3}.$$

With these values it is evident that (11a) is an identity.

Let the quantity Δ be defined by the relation

$$\Delta = \begin{vmatrix} L_1(W_1) & L_1(W_2) \\ L_2(W_1) & L_2(W_2) \end{vmatrix}. \tag{13}$$

Inasmuch as the constants $\bar{\mu}_{ij}$, $\bar{\nu}_{ij}$ can be expressed as polynomials in μ_{ij}, ν_{ij}, the same is true of Δ. We shall make the following hypothesis concerning the given equation, namely that

(vi) $$\Delta \not\equiv 0 \text{ in } \mu_{ij}, \nu_{ij}.$$

Then we may suppose these parameters chosen so that

$$\Delta \neq 0. \tag{14}$$

The function $\boldsymbol{K}(x, \xi)$ is defined now by the formula

$$\boldsymbol{K}(x,\xi) = \frac{1}{\Delta}\begin{vmatrix} K(x,\xi) & W_1(x) & W_2(x) \\ V_1(\xi) & L_1(W_1) & L_1(W_2) \\ V_2(\xi) & L_2(W_1) & L_2(W_2) \end{vmatrix}. \tag{15}$$

With the use of relation (11) it is readily verified that $\boldsymbol{K}(x, \xi)$ shares with $K(x, \xi)$ its characteristics (B) to (D), while

$$L_i(\boldsymbol{K}) \equiv M_i(\boldsymbol{K}) \equiv 0 \qquad (i = 1,2). \tag{16}$$

7. **The relation between $\boldsymbol{K}(x, \xi)$ and $G(x, \xi)$.** The function

$$\omega(x,\xi) = \boldsymbol{K}(x,\xi) - G(x,\xi)$$

is continuous together with its partial derivatives of second order in R, and satisfies the differential system

$$\frac{\partial^2\omega(x,\xi)}{\partial^2\xi} = \boldsymbol{K}_{\xi\xi}(x,\xi),$$

$$M_i(\omega) \equiv 0 \qquad (i = 1,2).$$

It follows* that we have for $G(x, \xi)$ as a function of x the integral equation

* Cf. paper 1, §7.

$$K(x,\xi) - G(x,\xi) = -\int_a^b E(x,t)G(t,\xi)dt, \tag{17}$$

where

$$E(x,t) = -K_{tt}(x,t).$$

From (15) we may write

$$K(x,t) = K(x,t) - \theta_1(x)V_1(t) - \theta_2(x)V_2(t), \tag{18}$$

where we observe in passing that

$$L_i(\theta_j) = \delta_{ij} \qquad (i,j = 1,2). \tag{19}$$

From (18) it is seen that the parameters μ_{ij}, ν_{ij} are contained in the kernel of (17) only rationally and only in such terms as are products of a function of x by a function of t. It can be shown because of this that the Fredholm determinant D and the first minor $D(x, t)$ of the kernel $E(x, t)$ are themselves rational functions of these parameters.* We shall make the hypothesis that for the given equation

$$D \not\equiv 0 \text{ in } \mu_{ij},\nu_{ij}, \tag{vii}$$

and shall suppose the parameters chosen so that

$$D \neq 0. \tag{20}$$

Then the reciprocal $F(x, t)$ of the kernel $E(x, t)$ exists and the relation (17) may be written in the "solved" form

$$G(x,t) = K(x,t) - \int_a^b F(x,t)K(t,\xi)dt. \tag{21}$$

8. **Properties of $F(x, t)$.** The kernel $E(x, t)$ and hence also the reciprocal are continuous in R. From (16) we obtain the relation

$$L_i(E) \equiv 0, \tag{22a}$$

and from Fredholm's identities

$$E(x,\xi) + F(x,\xi) = \int_a^b E(x,t)F(t,\xi)dt = \int_a^b F(x,t)E(t,\xi)dt \tag{23}$$

it follows further that

$$L_i(F) \equiv 0, \tag{22b}$$

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* Cf. the method of proof in paper 1, §8 or Bateman, H., *A formula for the solving function of a certain integral equation of the second kind*, Messenger of Mathematics, vol. 37(1907), p. 179.

and that $F_x(x, \xi)$ is possessed of partial derivatives to those of order $(n-3)$ which are continuous in R_1 and R_2. Lastly since $F(x, \xi) = D(x, \xi)/D$, this reciprocal is rational in the parameters μ_{ij}, ν_{ij}.

9. **Transformation of equation (7).** Let $u(x)$ now be any solution of the normalized equation (7). Then it follows from (7), (10) and (18) that

$$\lambda^2 \int_a^b V_i(\xi)u(\xi)d\xi = L_i(u) \tag{24}$$

and

$$\lambda^2 \int_a^b K(x,\xi)u(\xi)d\xi = u(x) - L_1(u)\theta_1(x) - L_2(u)\theta_2(x).$$

With these relations we find upon multiplying (21) by $\lambda^2 u(\xi)$ and integrating that

$$\lambda^2 \int_a^b G(x,\xi)u(\xi)d\xi = u(x) - L_1(u)\Phi_1(x) - L_2(u)\Phi_2(x) - \int_a^b F(x,t)u(t)dt, \tag{25}$$

where

$$\Phi_i(x) = \theta_i(x) - \int_a^b F(x,t)\theta_i(t)dt \qquad (i = 1,2). \tag{26}$$

We observe that $\Phi_i(x)$ possesses continuous derivatives to those of order $(n-2)$ and that

$$L_i(\Phi_j) = \delta_{ij} \qquad (i,j = 1,2). \tag{27}$$

Returning to relation (25) and differentiating it twice we obtain an integro-differential equation satisfied by $u(x)$. If we abbreviate by setting

$$\begin{aligned} \Phi_i''(x) &= \phi_i(x), \\ F_x(x,\xi)\Big]_{\xi=x}^{\xi=x+} &= g(x), \\ F_{xx}(x,\xi) &= f(x,\xi), \end{aligned} \tag{28}$$

and associate the equation last obtained with (24) above the result may be formulated as in the following theorem.

THEOREM 2. *If the kernel of the normalized integral equation is such that the conditions* (v), (vi), *and* (vii) *are satisfied then every solution of the integral equation is also a solution of the integro-differential system*

$$\text{(a)}\quad u''(x) - \lambda^2 u(x) = L_1(u)\phi_1(x) + L_2(u)\phi_2(x) + \int_a^b f(x,\xi)u(\xi)d\xi - g(x)u(x), \tag{29}$$

$$\text{(b)}\quad L_i(u) = \lambda^2 \int_a^b V_i(\xi)u(\xi)d\xi \qquad (i = 1,2).$$

Of the functions $g(x)$ and $\phi_i(x)$, the first possesses continuous derivatives to order $(n-3)$ while the others possess continuous derivatives to order $(n-4)$. The function $f(x, \xi)$ and its partial derivatives to order $(n-4)$ are continuous in the regions R_1 and R_2.

10. **The associated integral equation.** The equation associated with (7), i.e.

$$v(x) = \lambda^2 \int_a^b K(\xi, x) v(\xi) d\xi, \tag{$\overline{7}$}$$

is also in normal form if the partial derivatives of the kernel are properly defined on the line $\xi = x$. The deductions already made are easily adapted, therefore, to apply to this equation as well. Thus we may set $\overline{K}(x, \xi) = K(\xi, x)$, and let the two sets of parameters μ_{ij}, ν_{ij} and $\bar{\mu}_{ij}, \bar{\nu}_{ij}$ be interchanged. If we distinguish the various functions arising then in the consideration of ($\overline{7}$) by superscribing them with a bar, it is readily found that

$$\overline{W}_i(x) \equiv V_i(x), \qquad \bar{V}_i(\xi) \equiv W_i(\xi),$$

and, since L_i and M_i must be interchanged, that $\bar{\Delta} = \Delta$. The condition $\bar{\Delta} \not\equiv 0$ in the parameters is therefore already covered. It is, however, necessary to make the further hypothesis that

($\overline{\text{vii}}$) $$\bar{D} \not\equiv 0 \quad \text{in} \quad \bar{\mu}_{ij}, \bar{\nu}_{ij}.$$

The conclusions embodied in Theorem 2 for equation (7) can then be drawn in an analogous manner for equation ($\overline{7}$).

The substitution of the equation (5) for equation (7) in §3 can now be motivated. If the kernel $K(x, \xi)$ is orthogonal on (a, b) to a function $\sigma(x)$ it follows from formulas (15) and (10) that

$$\int_a^b K(x, \xi) \sigma(x) dx = 0$$

identically in μ_{ij}, ν_{ij}. But differentiating this equation twice it follows that

$$\sigma(\xi) = \int_a^b E(x, \xi) \sigma(x) dx,$$

and hence that $D \equiv 0$ in μ_{ij}, ν_{ij}. Similarly if $K(x, \xi)$ is orthogonal to $\omega(\xi)$, then $\overline{D} \equiv 0$. If the kernel of equation (5) is orthogonal both to functions of x and to functions of ξ, therefore, a discussion similar to that above but applied to the equation (5) or to its associated equation would necessarily be stopped by conditions (vii) and ($\overline{\text{vii}}$).

11. **The transformation of system (29).** The equivalence of system (29) and equation (7) will be established if it is shown that every solution of the former solves also the latter. Let us suppose the system (29) given. Then the functions $\Phi_i(x)$, $i=1, 2$, and $F(x, \xi)$ may be determined respectively as the unique continuous solutions of the differential systems

$$y''(x) = \phi_i(x),$$
$$L_j(y) = \delta_{ji} \qquad (i,j = 1,2),$$

and as the unique solution continuous in R_1 and R_2 of the system

$$\frac{\partial^2 Y(x,\xi)}{\partial x^2} = f(x,\xi),$$

$$L_i(Y) \equiv 0, \qquad Y_x(x,\xi)\Big]_{\xi-x}^{\xi-x+} = g(x).$$

Let $\sigma(x)$ be constructed now by the formula

$$\sigma(x) = u(x) - \lambda^2 \int_a^b G(x,\xi)u(\xi)d\xi - L_1(u)\Phi_1(x) - L_2(u)\Phi_2(x) - \int_a^b F(x,t)u(t)dt,$$

$u(x)$ being any solution of (29a). By (29a), IV, (22b), and (27), this function is a solution of the incompatible system (8). Hence it follows that $\sigma(x)\equiv 0$, namely that $u(x)$ is a solution of (25).

The function $E(x, \xi)$ is determined as the reciprocal of $F(x, \xi)$, and the functions $K(x, \xi)$ and $\theta_i(x)$ are respectively determined by relations (17) and (26).

Let (17) be multiplied by $\lambda^2 u(\xi)$ and integrated with respect to ξ. Eliminating

$$\lambda^2 \int_a^b G(x,\xi)u(\xi)d\xi$$

from the result by means of (25), we find because of (26) and (23) that every solution of (29a) is also a solution of the equation

$$u(x) = \lambda^2 \int_a^b K(x,\xi)u(\xi)d\xi + L_1(u)\theta_1(x) + L_2(u)\theta_2(x). \tag{30}$$

If the function $u(x)$ is a solution of the entire system (29) we may substitute further for $L_i(u)$ in (30) the values given by (29b). Because of (18) this results in reducing equation (30) to the form (7).

THEOREM 3. *Under the hypotheses of Theorem* 2 *every solution of the integro-differential system* (29) *is also a solution of the normalized integral equation* (7).

12. **The general system of type** (29). In the preceding section we were concerned with the system (29) which was deduced from equation (7). This fact affected the deductions only to the extent that the function $E(x, \xi)$ reciprocal to $F(x, \xi)$ was known to exist. In any case the final function is found from (21), (26) and (18) to satisfy the relation

$$(31)\quad K(x,\xi) = \{G(x,\xi) - V_1(\xi)\Phi_1(x) - V_2(\xi)\Phi_2(x)\} + \int_a^b F(x,t)K(t,\xi)dt,$$

which may be solved if $F(x, t)$ possesses a reciprocal. Hence we may state

THEOREM 4. *If the function* $F(x, \xi)$ *associated in the manner above with an integro-differential system of type* (29) *possesses a reciprocal, then the system is equivalent to an integral equation of the second kind.*

Clearly the equivalence of equation $(\overline{7})$ with the integro-differential system deduced from it may be established similarly under the hypotheses made.

CHAPTER 4. THE EXISTENCE OF A SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION

13. **Notation and lemmas.** We shall adopt now the following as generic symbols:

N to represent a positive constant sufficiently large;

$H(x, \xi, \lambda)$ to represent a function which is uniformly bounded for $|\lambda| > N$;

$\epsilon(x, \xi, \lambda)$ to represent a function which approaches zero uniformly as $|\lambda| \to \infty$;

$B(x, \xi, \lambda)$ to represent a function which for $|\lambda| > N$ is uniformly bounded and in integrable as to ξ uniformly in x and λ;*

$E(x, \lambda)$ to represent a function which is uniformly bounded for $|\lambda| > N$, and which is continuous in x and analytic in λ;

M to represent a constant which is positive or zero.

It is convenient also to have at hand the following lemmas.† In them σ denotes the real part of λ.

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* Cf. Langer and Tamarkin, *A notion of uniform integrability*, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 335.

† For proofs see paper 1, §17.

LEMMA 1. *If* $\sigma \leqq M$ *and* $a \leqq z \leqq b$, *then*

$$\int_a^z e^{\lambda(x-\xi)} B(x,\xi,\lambda)d\xi = \epsilon(x,z,\lambda).$$

LEMMA 2. *If* $\sigma \geqq -M$ *and* $a \leqq z \leqq b$, *then*

$$\int_b^z e^{\lambda(x-\xi)} B(x,\xi,\lambda)d\xi = \epsilon(x,z,\lambda).$$

14. Transformation of the integro-differential equation. Let $P(u, x)$ designate the right hand member of equation (29a) and consider the equation

$$u''(x) - \lambda^2 u(x) = P(u,x) + E(x,\lambda). \tag{32}$$

Looking upon it as a differential equation we may write

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + \frac{1}{2\lambda}\int_*^x [e^{\lambda(x-\xi)} - e^{-\lambda(x-\xi)}][P(u,\xi) + E(\xi,\lambda)]d\xi, \tag{32a}$$

the coefficients c_1 and c_2 being arbitrary but constant with respect to x, and $*$ representing a constant which may be chosen arbitrarily for each term of the integrand. Confining the attention to the case $\sigma \leqq M$ we assume for the moment that equation (32a) is possessed of a solution for the following values of the arbitrary elements:

$$c_1 = ke^{-\lambda a} \qquad (k \text{ a constant}),$$

$$c_2 = 0,$$

$* = a$ for terms of the integrand involving $e^{\lambda(x-\xi)}$,

$* = b$ for terms of the integrand involving $e^{-\lambda(x-\xi)}$.

With the abbreviations

$$\bar{\phi}_i(x) = \frac{1}{2}\int_a^b e^{\lambda|x-\xi|}\phi_i(\xi)d\xi,$$

$$\bar{E}(x) = \frac{1}{2}\int_a^b e^{\lambda|x-\xi|}E(x,\xi)d\xi,$$

$$\bar{f}(x,\xi) = \frac{1}{2}\left\{\int_a^b f(t,\xi)e^{\lambda|x-t|}dt - g(\xi)e^{\lambda|x-\xi|}\right\},$$

the equation may then be written

$$u(x) = ke^{\lambda(x-a)} + \frac{1}{\lambda}\left\{L_1(u)\bar{\phi}_1(x) + L_2(u)\bar{\phi}_2(x) + \bar{E}(x) + \int_a^b \bar{f}(x,\xi)u(\xi)d\xi\right\}. \tag{32b}$$

Looking upon this as an explicit evaluation of $u(x)$ and forming from it the expressions $L_i(u)$, we obtain

$$L_i(u) = L_1(u)\Phi_{i1} + L_2(u)\Phi_{i2} + T_i + \int_a^b R_i(\xi)u(\xi)d\xi \qquad (i = 1,2), \tag{33}$$

where

$$\Phi_{ij} = c_{i1}\bar{\phi}_j(a) + c_{i2}\bar{\phi}_j(b), \qquad c_{i1} = \frac{\mu_{i1}}{\lambda} - \mu_{i2},$$

$$R_i(\xi) = c_{i1}\bar{f}(a,\xi) + c_{i2}\bar{f}(b,\xi), \qquad c_{i2} = \frac{\nu_{i1}}{\lambda} + \nu_{i2},$$

$$T_i = (\mu_{i1} + \lambda\mu_{i2})k + \lambda c_{i2}ke^{\lambda(b-a)} + c_{i1}\overline{E}(a) + c_{i2}\overline{E}(b).$$

The equations (33) compose an algebraic system for the quantities $L_i(u)$ with determinant

$$C = \begin{vmatrix} 1 - \Phi_{11} & -\Phi_{12} \\ -\Phi_{21} & 1 - \Phi_{22} \end{vmatrix}.$$

By Lemmas 1 and 2, $\bar{\phi}_i(x) = \epsilon(x, \lambda)$, and hence $\Phi_{ij} = \epsilon(\lambda)$ and $C^{-1} = H(\lambda)$. Hence the system (33) may be solved if $|\lambda| > N$. If we denote by γ_{ij} the minor of the element in the ith row and jth column of C, the solution is

$$L_i(u) = C^{-1}\sum_{j=1}^{2}\gamma_{ji}\left[T_j + \int_a^b R_j(\xi)u(\xi)d\xi\right]. \tag{33a}$$

With these values equation (32b) becomes

$$u(x) = \theta(x,\lambda) + \int_a^b \Omega(x,\xi,\lambda)u(\xi)d\xi, \tag{32c}$$

where

$$\theta(x,\lambda) = ke^{\lambda(x-a)} + \frac{\overline{E}(x)}{\lambda} + \frac{C^{-1}}{\lambda}\sum_{i,j=1}^{2}\bar{\phi}_i(x)\gamma_{ji}T_j, \tag{34}$$

and

$$\Omega(x,\xi,\lambda) = \frac{1}{\lambda}\left\{\bar{f}(x,\xi) + C^{-1}\sum_{i,j=1}^{2}\bar{\phi}_i(x)\gamma_{ji}R_j(\xi)\right\}. \tag{35}$$

15. **The existence of a solution of equation (32).** Returning now to the equation (32) let the functions $\theta(x, \lambda)$ and $\Omega(x, \xi, \lambda)$ be constructed from its

coefficients by the formulas above. It is readily found then that for $\sigma \leqq M$ and $|\lambda| > N$, $\theta(x, \lambda)$, $\theta_x(x, \lambda)/\lambda$, $\Omega_x(x, \xi, \lambda)/\lambda$ are bounded, while

$$|\Omega(x,\xi,\lambda)| < \frac{1}{2(b-a)}.$$

Hence the equation (32c) possesses a solution $u(x)$, which satisfies the relations $|u(x)| < A$, $|u'(x)/\lambda| < A'$, and which is analytic in λ for $|\lambda| > N$. To show that $u(x)$ satisfies also equation (32) it is necessary only to reverse the deductions above.

The case $\sigma \geqq -M$ may be treated similarly to that above, the arbitrary elements in (32a) being chosen as follows:

$$c_1 = 0, \qquad c_2 = ke^{\lambda a},$$

$$* = b \text{ for terms involving } e^{\lambda(x-\xi)},$$

$$* = a \text{ for terms involving } e^{-\lambda(x-\xi)}.$$

The results also are similar to those there obtained.

If the function $E(x, \lambda)$ vanishes identically the equation considered coincides with (29a) and we may summarize our results as follows.

THEOREM 5. *If the parameter λ is restricted to any region bounded by a line parallel to the axis of imaginaries and exterior to a circle sufficiently large with center at $\lambda = 0$ then the equation* (29a) *admits of a solution $u(x)$ which is analytic in λ and satisfies the relations*

$$|u(x)| < A \text{ (a constant)}, \quad \left|\frac{u'(x)}{\lambda}\right| < A' \text{ (a constant)}.$$

If $E(x, \lambda) \not\equiv 0$, the choice $k = 0$ is possible. If we assume further that $E(x, \lambda)$ is integrable in x uniformly with respect to λ, then by Lemmas 1 and 2, $\overline{E}(x) = \epsilon(x, \lambda)$, and with $k = 0$, $\theta(x, \lambda) = \epsilon(x, \lambda)/\lambda$. In this case we have the

THEOREM 6. *If the parameter λ is confined to a region as described in Theorem* 5 *and if the function $E(x, \lambda) \not\equiv 0$ is integrable in x uniformly with respect to λ, then equation* (32) *admits of a solution $u(x)$ which is analytic in λ and which is such that*

$$u(x) = \frac{\epsilon(x,\lambda)}{\lambda}, \qquad u'(x) = \epsilon(x,\lambda).$$

CHAPTER 5. THE SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION

16. **The probable form of a solution.** To derive the probable form of a solution of equation (29a) we consider the scheme of successive approximation defined by the formula

$$u_j''(x) - \lambda^2 u_j(x) = P(u_{j-1}, x),$$

where $P(u, x)$ is the right hand member of (29a). A particular function $u_j(x)$ in terms of $u_{j-1}(x)$ is given by

$$u_j(x) = e^{\lambda x} + \frac{1}{2}\int_*^x [e^{\lambda(x-t)} - e^{-\lambda(x-t)}]P(u_{j-1}, t)dt, \tag{36}$$

with any constant choice of $*$ for each term of the integrand. The initial function $u_0(x) \equiv 0$ leads to $u_1(x) = e^{\lambda x}$. With the use of Lemmas 1 and 2 and with proper choice of $*$ from the values $*=a$, $*=b$ and $*=\frac{1}{2}(a+b)$ it is found that for $j=2$, (36) admits of a solution of the form

$$u_2(x) = e^{\lambda a}H(x,\lambda) + e^{\lambda b}H(x,\lambda) + e^{\lambda x}H(x,\lambda) + e^{\lambda(b+a-x)}H(x,\lambda).$$

In the further repetition of the process the introduction of terms of a type not already represented is avoidable. We are thus led to expect the existence of a solution with the functional form of $u_2(x)$ above.

17. **The formal determination of coefficients.** We shall write the functions $H(x, \lambda)$ now as series in descending powers of λ, and seek to evaluate the coefficients. With the notation

$$[\omega(x)] = \sum_{j=0}^{\infty} \frac{\omega_j(x)}{\lambda^j},$$

we have then

$$u(x) = e^{\lambda x}[\gamma(x)] + e^{\lambda(b+a-x)}[\delta(x)] + e^{\lambda b}[\beta(x)] + e^{\lambda a}[\alpha(x)]. \tag{37}$$

Substituting this into equation (29a) and integrating by parts the terms involving $f(x, \xi)[\gamma(\xi)]$ and $f(x, \xi)[\delta(\xi)]$ we obtain the result

$$e^{\lambda x}\Big\{ 2\lambda[\gamma'(x)] + [\gamma''(x)] + g(x)[\gamma(x)] - \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}} \sum_{j=0}^{i} (-1)^j \frac{\partial}{\partial \xi^j}[f(x,\xi)\gamma_{i-j}(\xi)]\Big]_{\xi=x}^{\xi=x+} \Big\}$$
$$+ e^{\lambda(b+a-x)}\{\ \} + e^{\lambda b}\{\ \} + e^{\lambda a}\{\ \} = 0.$$

Equating to zero the coefficient of $e^{\lambda x}/\lambda^l$ for $l=-1, 0, 1, 2, \cdots$, we obtain equations from which the quantities $\gamma_i(x)$ may be successively determined. In particular the choice $\gamma_0=1$ is possible. Further equating to zero the coefficient of $e^{\lambda(b+a-x)}/\lambda^l$ we find as a possible choice $\delta_i(x)\equiv 0$ for all i, while the terms in $e^{\lambda b}/\lambda^l$ and $e^{\lambda a}/\lambda^l$ similarly yield formulas for the determination of the quantities $\beta_i(x)$ and $\alpha_i(x)$. With the values already chosen we find in particular $\beta_0=\alpha_0=0$.

18. **The function $\bar{u}(x)$.** The extent to which the formal process thus developed is actually applicable is limited by the number of derivatives which the coefficients of equation (29a) admit. Referring to §9, therefore, and considering the formulas obtained, we find that $\gamma_i(x)$ possesses $(n-1-i)$ derivatives while $\beta_{2i-1}(x)$, $\beta_{2i}(x)$, $\alpha_{2i-1}(x)$, $\alpha_{2i}(x)$ possess derivatives of order $(n-2-2i)$. The last of these coefficients which may be differentiated twice are therefore $\gamma_{n-3}(x)$, $\beta_{n-4}(x)$, $\alpha_{n-4}(x)$ if n is even and $\gamma_{n-3}(x)$, $\beta_{n-5}(x)$, $\alpha_{n-5}(x)$, if n is odd. To avoid duplication in the discussion we shall suppose that for the case in hand n is even. Then the function

$$\begin{aligned}(38)\qquad \bar{u}(x) = e^{\lambda x}&\left\{1+\frac{\gamma_1(x)}{\lambda}+\cdots+\frac{\gamma_{n-3}(x)}{\lambda^{n-3}}\right\}\\ &+e^{\lambda b}\left\{\frac{\beta_1(x)}{\lambda}+\cdots+\frac{\beta_{n-4}(x)}{\lambda^{n-4}}\right\}\\ &+e^{\lambda a}\left\{\frac{\alpha_1(x)}{\lambda}+\cdots+\frac{\alpha_{n-4}(x)}{\lambda^{n-4}}\right\}\end{aligned}$$

is twice differentiable. It is readily seen to satisfy an equation of the form

$$(39)\qquad \bar{u}''(x)-\lambda^2\bar{u}(x)=P(\bar{u},x)+\frac{1}{\lambda^{n-5}}\{e^{\lambda b}B(x,\lambda)+e^{\lambda a}B(x,\lambda)\},$$

the functions $B(x, \lambda)$ being analytic in λ for $|\lambda|>N$.

19. **The true solutions.** If $u(x)$ is any solution of equation (29a) the function $u(x)-\bar{u}(x)$ is a solution of equation (39) and conversely. On the other hand if equation (39) is multiplied by $\lambda^{n-5}e^{-\lambda a}$ it takes, for $\sigma\leqq M$, the form (32) with

$$E(x,\lambda)=e^{\lambda(b-a)}B(x,\lambda)+B(x,\lambda).$$

This function $E(x, \lambda)$ is integrable in x uniformly with respect to λ. It follows by Theorem 6 that this modified equation possesses a solution of the form $\epsilon(x, \lambda)/\lambda$, with derivatives of the form $\epsilon(x, \lambda)$. Hence we conclude that for $\sigma\leqq M$ equation (29a) possesses a solution given by

$$u(x) - \bar{u}(x) = \frac{\epsilon(x,\lambda)e^{\lambda a}}{\lambda^{n-4}},$$

$$u'(x) - \bar{u}'(x) = \frac{\epsilon(x,\lambda)e^{\lambda a}}{\lambda^{n-5}}.$$

For $\sigma \geqq -M$ a similar argument may be applied to equation (39) multiplied by $\lambda^{n-5}e^{-\lambda b}$. Hence we conclude that equation (29a) possesses for $\sigma \geqq -M$ a solution given by

$$u(x) - \bar{u}(x) = \frac{\epsilon(x,\lambda)e^{\lambda b}}{\lambda^{n-4}},$$

$$u'(x) - \bar{u}'(x) = \frac{\epsilon(x,\lambda)e^{\lambda b}}{\lambda^{n-5}}.$$

It is clear that we may summarize these results as follows.

THEOREM 7. *If λ is confined to a region of the type described in Theorem 5 then the integro-differential equation* (29a) *admits of a solution $u_1(x)$ of the form*

$$\begin{aligned} u_1(x) = {} & e^{\lambda x}\left\{1 + \frac{\gamma_1(x)}{\lambda} + \cdots + \frac{\gamma_{n-4}(x)}{\lambda^{n-4}}\right\} \\ & + e^{\lambda b}\left\{\frac{\beta_1(x)}{\lambda} + \cdots + \frac{\beta_{n-4}(x) + \epsilon(x,\lambda)}{\lambda^{n-4}}\right\} \\ & + e^{\lambda a}\left\{\frac{\alpha_1(x)}{\lambda} + \cdots + \frac{\alpha_{n-4}(x) + \epsilon(x,\lambda)}{\lambda^{n-4}}\right\}, \end{aligned} \tag{40}$$

the first derivatives of the functions $\epsilon(x, \lambda)$ involved being of the form $\lambda\epsilon(x, \lambda)$.

The equation (29a) involves λ only to the even power two. It follows that we have a second solution of this equation, namely

$$u_2(x,\lambda) = u_1(x, -\lambda). \tag{41}$$

Clearly these two solutions are linearly independent.

20. **The associated equation.** Inasmuch as this chapter and the preceding one involve no new hypotheses concerning the equation (7), it is clear that a precisely analogous discussion of the associated equation $(\bar{7})$ is possible. The corresponding integro-differential equation admits, therefore, of solutions $v_1(x)$, $v_2(x)$ whose form may be obtained from $u_1(x)$ and $u_2(x)$ by replacing γ_i, β_i and α_i, by $\bar{\gamma}_i$, $\bar{\beta}_i$ and $\bar{\alpha}_i$ respectively.

CHAPTER 6. THE CHARACTERISTIC VALUES

21. **The characteristic equation.** Since the equation (29a) is linear we have as a solution the expression

$$u(x) = c_1 u_1(x) + c_2 u_2(x), \tag{42}$$

where c_1 and c_2 are arbitrary constants. By Theorems 2 and 3, such a solution will satisfy the given integral equation if and only if it satisfies also the equations (29b). Setting

$$\omega_{ij}(\lambda) = L_i(u_j) - \lambda^2 \int_a^b V_i(\xi) u_j(\xi) d\xi \qquad (i,j = 1,2), \tag{43}$$

and substituting in (29b) the form (42) we obtain as the relations to be satisfied

$$c_1\omega_{i1}(\lambda) + c_2\omega_{i2}(\lambda) = 0 \qquad (i = 1,2). \tag{44}$$

These equations compose a homogeneous system for the constants c_1 and c_2 and admit of a solution if

$$\begin{vmatrix} \omega_{11}(\lambda) & \omega_{12}(\lambda) \\ \omega_{21}(\lambda) & \omega_{22}(\lambda) \end{vmatrix} = 0. \tag{45}$$

This equation in λ we shall call the characteristic equation. Its roots, which are the values of λ for which (42) includes a solution of the integral equation, are the characteristic values.

To deduce the functional form of (45) we substitute for $\lambda^2 u_j(\xi)$ in (43) its value as the second term in (29a). Upon integrating twice by parts the resulting term in $u_j''(\xi)$ we obtain the formula

$$\omega_{i1}(\lambda) = k_{i1}u_1(a) + k_{i2}u_1(b) + k_{i3}u_1'(a) + k_{i4}u_1'(b) - \int_a^b \Omega_i(\xi) u_1(\xi) d\xi, \tag{43a}$$

where

$$k_{i4} = \nu_{i1} - V_i(b) + \nu_{12}\int_a^b V_i(\xi)\phi_1(\xi)d\xi + \nu_{22}\int_a^b V_i(\xi)\phi_2(\xi)d\xi \qquad (i = 1,2), \tag{46}$$

and k_{ij}, $j = 1, 2, 3$, are given by similar forms, and where

$$\Omega_i(\xi) = V_i''(\xi) + V_i(\xi)g(\xi) - \int_a^b f(t,\xi)V_i(t)dt \qquad (i = 1,2).$$

Substituting in (43a) the forms

$$\begin{aligned} u_1(x) &= e^{\lambda x} + e^{\lambda b}\epsilon(x,\lambda) + e^{\lambda a}\epsilon(x,\lambda), \\ u_1'(x) &= \lambda\{e^{\lambda x} + e^{\lambda b}\epsilon(x,\lambda) + e^{\lambda a}\epsilon(x,\lambda)\}, \end{aligned} \tag{40a}$$

obtained from Theorem 7, we find that

$$\omega_{i1}(\lambda) = \lambda e^{\lambda a}\{k_{i3} + \epsilon(\lambda)\} + \lambda e^{\lambda b}\{k_{i4} + \epsilon(\lambda)\}. \tag{43b}$$

For the reductions here involved it is convenient to use the results of the following lemma.

LEMMA 3*. *If z_1 and z_2 are any points of the interval (a, b), then*

$$\int_{z_1}^{z_2} e^{\lambda t} H(\xi, t, \lambda)\, dt = e^{\lambda z_2} H(z_2, \xi, \lambda) + e^{\lambda z_1} H(z_1, \xi, \lambda).$$

From (41) and (43) it is readily seen that $\omega_{i2}(\lambda) = \omega_{i1}(-\lambda)$. Hence (43b) yields for the characteristic equation the form

$$\begin{vmatrix} k_{13} & k_{14} \\ k_{23} & k_{24} \end{vmatrix} [e^{\lambda(b-a)} - e^{-\lambda(b-a)}] + e^{\lambda(b-a)}\epsilon(\lambda) + \epsilon(\lambda) + e^{-\lambda(b-a)}\epsilon(\lambda) = 0.$$

The quantities k_{ij} are rational functions of the parameters μ_{ij}, ν_{ij}, as is seen from formula (46) etc. The vanishing or non-vanishing character of the determinant $(k_{13}\, k_{24} - k_{14}\, k_{23})$ must, however, be independent of the choice of these parameters, since it vitally affects the structure of the characteristic equation, which depends only on the integral equation given.† We shall make the hypothesis that the given integral equation is such that

(viii) $$\begin{vmatrix} k_{13} & k_{14} \\ k_{23} & k_{24} \end{vmatrix} \neq 0.‡$$

The characteristic equation may then be written, for $|\lambda| > N$, in the form

(45a) $$e^{2\lambda(b-a)} + \epsilon(\lambda) e^{\lambda(b-a)} - [1 + \epsilon(\lambda)] = 0.$$

22. **The characteristic values.** The equation (45a) is formally a quadratic in $e^{\lambda(b-a)}$. Factoring it we obtain

$$e^{\lambda(b-a)} = 1 + \epsilon(\lambda), \qquad e^{\lambda(b-a)} = -1 + \epsilon(\lambda).$$

From this it follows that

(45b)
$$\lambda = \frac{2m\pi i}{b-a} + \epsilon(\lambda),$$
$$\lambda = \frac{(2m+1)\pi i}{b-a} + \epsilon(\lambda) \quad (m = \pm M, \pm (M+1), \cdots).$$

It is readily shown§ that for $|m|$ sufficiently large these equations admit respectively just one root in the neighborhood of $2m\pi i/(b-a)$, and $(2m+1)\pi i/(b-a)$. These are the characteristic values.

* For the proof of this lemma see paper 1, §23.

† The analogous situation arising in paper 1, §33, is illustrated in paper 1, appendix.

‡ A case in which this condition is not met is discussed in chapter 10.

§ Cf. paper 1, §33.

By the classical theory of integral equations, associated equations have the same characteristic values. We observe, therefore, that our results are obtainable by the methods outlined when either equation (7) or equation ($\overline{7}$) satisfies the hypotheses made. If we designate by λ_m the characteristic value in the neighborhood of $m\pi i/(b-a)$, we may formulate the result as follows:

THEOREM 8. *Under the hypotheses* (v), (vi) *and* (viii) *and either* (vii) *or* ($\overline{\text{vii}}$), *the normalized integral equation* (7) *admits of infinitely many characteristic values. For* $|m|$ *sufficiently large these values are given by the formula*

$$\lambda_m = \frac{m\pi i}{b-a} + \epsilon_m \quad (m = \pm M, \pm (M+1), \cdots), \tag{47}$$

where $\lim_{|m|\to\infty} \epsilon_m = 0$.

23. **A more precise formula for λ_m.** The deductions just made if carried out with the assumption of a greater number of derivatives of the kernel of the integral equation will lead to a more precise formula. Thus if $n \geqq 6$, then by Theorem 7 there exists a function $u_1(x)$ given by the theorem for $n=6$. With this form it is found that

$$\begin{aligned}\omega_{i1}(\lambda) = {} & \lambda e^{\lambda a}\left\{ k_{i3} + \frac{k_{i3}\gamma_1(a) + k_{i1}}{\lambda} + \frac{H(\lambda)}{\lambda^2} \right\} \\ & + \lambda e^{\lambda b}\left\{ k_{i4} + \frac{k_{i4}\gamma_1(b) + k_{i2}}{\lambda} + \frac{H(\lambda)}{\lambda^2} \right\},\end{aligned} \tag{43c}$$

and hence that the functions $\epsilon(\lambda)$ in (45a) are of the type $A/\lambda + (\overline{A} + \epsilon(\lambda))/\lambda^2$, where A and $\overline{A}$ are constants. This leads to a formula

$$\lambda_{2m} = \frac{2m\pi i}{b-a} + \frac{A}{2m} + \frac{\overline{A} + \epsilon_{2m}}{(2m)^2},$$

with a similar form with distinct constants for λ_{2m+1}. We observe now, however, that since the given integral equation involves λ to an even power, $-\lambda_m$ is also a characteristic value. Clearly we may conclude from Theorem 8 that for $|m|$ sufficiently large $-\lambda_m = \lambda_{-m}$. Hence it follows that in the formula above $\overline{A} = 0$. We have, therefore,

THEOREM 9. *If, for the integral equation considered in Theorem* 8, $n \geqq 6$, *then the characteristic values are represented by a formula*

$$\lambda_m = \frac{m\pi i}{b-a} + \frac{A_1 + (-1)^m A_2}{m} + \frac{\epsilon_m}{m^2}, \tag{47a}$$

where A_1 *and* A_2 *are constants and* $\lim_{|m|\to\infty} \epsilon_m = 0$.

CHAPTER 7. THE CHARACTERISTIC FUNCTIONS

24. The solutions of the integral equation. The values λ_m have been determined so that for suitable choice of the constants c_1 and c_2 formula (42) yields a solution of the integral equation. From (44) we find, moreover, because of (45) that

$$c_1 : c_2 = -\left[h_1\omega_{12}(\lambda_m) + h_2\omega_{22}(\lambda_m)\right] : \left[h_1\omega_{11}(\lambda_m) + h_2\omega_{21}(\lambda_m)\right], \tag{48}$$

with any choice of the constant multipliers h_1 and h_2. We shall choose these multipliers so that

$$\begin{aligned} h_1k_{14} + h_2k_{24} &= 1, \\ h_1k_{13} + h_2k_{23} &= 0, \end{aligned} \tag{49}$$

which is possible because of (viii). If we observe then that $e^{\lambda_m x}$ is uniformly bounded we find from (49) with the use of (43b) that

$$c_1 : c_2 = \left[e^{-\lambda_m b} + \epsilon(\lambda_m)\right] : \left[e^{\lambda_m b} + \epsilon(\lambda_m)\right],$$

and hence from (42) with the use of (40a) that

$$u(x,\lambda_m) = e^{\lambda_m(b-x)} + e^{-\lambda_m(b-x)} + \epsilon(x,\lambda_m).$$

Substituting for λ_m its value as given by (47) this reduces to

$$u(x,\lambda_m) = 2\cos\frac{m\pi(b-x)}{b-a} + \epsilon(x,m).$$

25. The associated equation. The characteristic values of the equation (7) being also those of the associated equation $(\bar{7})$, the considerations of this and the preceding chapter apply equally well to either equation if both satisfy the hypotheses made. Under this assumption we may conclude, therefore, that the equation $(\bar{7})$ admits of a solution $v(x, \lambda_m)$ of precisely the form of $u(x, \lambda_m)$ above. Integrating the product of these solutions we find that

$$\int_a^b u(x,\lambda_m)v(x,\lambda_m)dx = 2(b-a) + \epsilon(m).$$

The solutions may, therefore, be normalized by multiplying them by

$$[2(b-a) + \epsilon(m)]^{-1/2}.$$

Since the given equation involves only λ^2 it follows that the solution for λ_m is identically that for λ_{-m}. Setting $u_m(x) = u(x, \lambda_m) = u(x, \lambda_{-m})$, $v_m(x) = v(x, \lambda_m) = v(x, \lambda_{-m})$, we may summarize our results as follows:

THEOREM 10. *Under the hypotheses* (v), (vi), (vii), ($\overline{\text{vii}}$) *and* (viii) *the normalized characteristic functions of the given integral equation and its associated equation are given respectively for m sufficiently large by the formulas*

$$(50)\qquad \begin{aligned} u_m(x) &= \left(\frac{2}{b-a}\right)^{1/2} \cos\frac{m\pi(b-x)}{b-a} + \epsilon(x,m), \\ v_m(x) &= \left(\frac{2}{b-a}\right)^{1/2} \cos\frac{m\pi(b-x)}{b-a} + \epsilon(x,m). \end{aligned}$$

26. **The more precise formulas.** If for the equation in hand $n \geqq 6$, more explicit formulas may be deduced by the method above. Thus we obtain from (48) and (43c)

$$c_1 : c_2 = \left[e^{-\lambda_m b} - \frac{Ae^{-\lambda_m b} + \overline{A}e^{-\lambda_m a}}{\lambda_m} + \frac{H(\lambda_m)}{\lambda_m^2}\right] : \left[e^{\lambda_m b} + \frac{Ae^{\lambda_m b} + \overline{A}e^{\lambda_m a}}{\lambda_m} + \frac{H(\lambda_m)}{\lambda_m^2}\right],$$

and using the form of $u_1(x, \lambda_m)$ given by Theorem 7 for $n = 6$, we obtain

$$u(x,\lambda_m) = \left[e^{\lambda_m(b-x)} + e^{-\lambda_m(b-x)}\right] + \frac{A - \gamma_1(x)}{\lambda_m}\left[e^{\lambda_m(b-x)} - e^{-\lambda_m(b-x)}\right] - \frac{\overline{A}}{\lambda_m}\left[e^{\lambda_m(x-a)} - e^{-\lambda_m(x-a)}\right] + \frac{H(\lambda_m)}{\lambda_m^2}.$$

Substituting in this the value of λ_m as given by (47a) and observing that

$$(-1)^m \sin\frac{m\pi(b-x)}{b-a} = -\sin\frac{m\pi(x-a)}{b-a},$$

we obtain the result

$$u(x,\lambda_m) = 2\cos\frac{m\pi(b-x)}{b-a} + \frac{H(x)}{m}\sin\frac{m\pi(b-x)}{b-a} + \frac{H(x)}{m}\sin\frac{m\pi(x-a)}{b-a} + \frac{H(x,m)}{m^2}.$$

Precisely as in §27 we conclude that the associated equation admits of a solution which is of the form $u(x, \lambda_m)$ above with distinct functions H. Moreover

$$\int_a^b u(x,\lambda_m)v(x,\lambda_m)dx = 2(b-a) + \frac{H(m)}{m^2}.$$

Hence we have the following theorem.

THEOREM 11. *If for the equation considered in Theorem* 10, $n \geqq 6$, *the normalized solutions of the equation and its associated equation are given for m sufficiently large by the formulas*

$$u_m(x) = \left(\frac{2}{b-a}\right)^{1/2} \cos\frac{m\pi(b-x)}{b-a} + \frac{1}{m}\left[H_1(x)\sin\frac{m\pi(b-x)}{b-a} + H_2(x)\sin\frac{m\pi(x-a)}{b-a}\right] + \frac{H(x,m)}{m^2},$$

(50a)

$$v_m(x) = \left(\frac{2}{b-a}\right)^{1/2} \cos\frac{m\pi(b-x)}{b-a} + \frac{1}{m}\left[\overline{H}_1(x)\sin\frac{m\pi(b-x)}{b-a} + \overline{H}_2(x)\sin\frac{m\pi(x-a)}{b-a}\right] + \frac{H(x,m)}{m^2}.$$

CHAPTER 8. THE CLOSURE OF THE SET OF CHARACTERISTIC FUNCTIONS

27. **Application of a theorem of Birkhoff.** We shall assume now that for the given integral equation

(va) $$n \geqq 6.$$

To show that the sets $u_m(x)$ and $v_m(x)$ are closed we shall employ the following extension of a theorem of Birkhoff.

Theorem. Let $\{u_m(x),\ v_m(x)\}$, $m=0,\ 1,\ 2,\cdots$, be a normalized biorthogonal system of continuous functions and $\{\bar{u}_m(x)\ \bar{v}_m(x)\}$ be a second biorthogonal normalized system which is such that (a) the series

$$\sum_{m=0}^{\infty} [u_m(x) - \bar{u}_m(x)]v_m(y), \qquad a \leqq x, y \leqq b,$$

converges uniformly to a function $H(x, y)$ less than $1/(b-a)$ in absolute value, and (b) the convergence is such that the series upon being multiplied by any continuous function may be integrated term by term as to x to yield a uniformly convergent series. Then if the set $\{u_m(x)\}$ involves a k-fold lack of closure, the lack of closure of the set $\{\bar{u}_m(x)\}$ is at most k-fold.*

28. **The integro-differential system with an additional parameter.** Consider the system

$$u''(x) - \lambda^2 u(x) = \eta P(u, x),$$

(51)

$$L_i(u) = \eta\lambda^2 \int_a^b V_i(\xi)u(\xi)d\xi \qquad (i = 1,2).$$

* Langer, Bulletin of the American Mathematical Society, vol. 33 (1927), p. 97.

For $\eta=1$ this is system (29). For $\eta=0$ it is, if we set $u(x, 0)=y(x)$, the differential system

$$\begin{aligned} y''(x) - \lambda^2 y(x) &= 0, \\ L_i(y) &= 0, \end{aligned} \tag{52}$$

which is equivalent to the integral equation

$$u(x,0) = \lambda^2 \int_a^b G(x,\xi)u(\xi,0)d\xi.$$

We may suppose the parameters μ_{ij}, ν_{ij} chosen subject to previous restrictions so that

$$\begin{vmatrix} \mu_{12} & \nu_{12} \\ \mu_{22} & \nu_{22} \end{vmatrix} \neq 0,^* \tag{53}$$

and also so that the system (52) has only simple characteristic values. With such a choice the solutions $u_m(x, 0)$ of system (52) and the solutions $v_m(x, 0)$ of the adjoint system compose a set of functions which is biorthogonal, normalized and closed. We wish to show that for η otherwise suitably chosen in the circle C: $|\eta| \leqq 1$ the system (51) still defines a set of solutions $u_m(x, \eta)$, and that there exists an associated biorthogonal normalized set $v_m(x, \eta)$.

The system (51) is obtained formally from (29) by replacing $f(x, \xi)$, $g(x)$, $\phi_i(x)$, and $V_i(\xi)$, $i=1, 2$, respectively by these functions multiplied by η. This has the effect of replacing $F(x, \xi)$ in chapter 3 by $\eta F(x, \xi)$, and $\Phi_i(x)$, $i=1, 2$, by $\Phi_i(x, \eta)$, polynomials of the first degree in η. If the neighborhoods of a finite number of critical points are excluded from the circle C and the remaining region is denoted by C', then it can be shown that for η in C' the following facts obtain.†

(a) The function $\eta F(x, \xi)$ possesses a reciprocal $E(x, \xi, \eta)$ which is analytic in η. Hence by Theorem 4 the system (51) is equivalent to an integral equation

$$u(x,\eta) = \lambda^2 \int_a^b K(x,\xi,\eta)u(\xi,\eta)d\xi, \tag{54}$$

in which the kernel is analytic in η.

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* This is the condition that system (52) be regular. Cf. Birkhoff, *Boundary value and expansion problems*, these Transactions, vol. 9 (1908), p. 383.

† Paper 1, §39.

(b) The integral equation associated with (54) is equivalent to an integro-differential system of the type considered in the foregoing chapters.

(c) $k_{13}(\eta)k_{24}(\eta) - k_{14}(\eta)k_{23}(\eta) \neq 0$,

where $k_{ji}(\eta)$ is given by formulas (46) etc. with the formal introduction of η as noted above.

Hence by chapters 6 and 7 the characteristic values $\lambda_m(\eta)$ exist and for m sufficiently large the solutions of (51) are given by formulas (50a), the functions $H(x)$ and $H(x, m)$ being replaced by functions $H(x, \eta)$ and $H(x, m, \eta)$, which are continuous in x and η.

Since the asserted facts have been established for the system (29) the region C' may be chosen to include the value $\eta = 1$. Moreover, since

$$k_{13}(0)k_{24}(0) - k_{14}(0)k_{23}(0) = \mu_{12}\nu_{22} - \mu_{22}\nu_{12},$$

from formulas (46) etc. it follows from (52) that $\eta = 0$ may also be included in C'.

29. **The application of the theorem of §27.** Let Γ represent now any closed curve connecting the points $\eta = 0$ and $\eta = 1$ and lying entirely within the region C', and let the values of η hereafter considered be restricted to this curve Γ. Using the symbol $\delta h(x)$ when $h(x, \eta)$ is any function of x and η to designate the difference in the value of h for $\eta = \bar{\eta}$ and $\eta = \eta$, i.e.

$$h(x,\bar{\eta}) - h(x,\eta) = \delta h(x),$$

we obtain from the formulas (50a) with η introduced as noted under (c) above the relation

$$\begin{aligned}\sum_{m=M}^{\infty} [u_m(x,\bar{\eta}) - u_m(x,\eta)]v_m(y,\bar{\eta}) &= \delta H(x) \sum_{m=M}^{\infty} \frac{\sin \dfrac{m\pi(b-x)}{b-a} \cos \dfrac{m\pi(b-y)}{b-a}}{m} \\ &+ \delta H(x) \sum_{m=M}^{\infty} \frac{\sin \dfrac{m\pi(x-a)}{b-a} \cos \dfrac{m\pi(b-y)}{b-a}}{m} \\ &+ \sum_{m=M}^{\infty} \frac{H(y,m)\delta H(x,m)}{m^2}.\end{aligned} \tag{55}$$

Of the series occurring here on the right the last converges uniformly to a value which becomes uniformly small with $|\bar{\eta} - \eta|$. Each of the remaining series may be transformed into a pair of series of the form $\sum_{m=M}^{\infty} (\sin m\pi z)/m$. Such series are known to converge everywhere uniformly save in the immediate vicinity of $z = 0, \pm 2, \pm 4, \cdots$, and the sum of any number of terms

is uniformly bounded. It follows* that the right hand member of (55) converges in the manner demanded by the theorem of §29, provided that $|\bar{\eta}-\eta|<\delta$, where δ is a positive constant chosen sufficiently small. Since the set $\{u_m(x, 0)\}$, $m=0, 1, 2, \cdots$, is known to be closed, the sub-set for $m=M, M+1, \cdots$, involves an M-fold lack of closure. It follows, therefore, first, that the lack of closure of the set $\{u_m(x, \eta)\}$, $m=M, M+1, \cdots$, for $|\eta|<\delta$ is at most M-fold, and subsequently, after a finite number of repeated applications of the theorem, that the set $\{u_m(x, 1)\}$, $m=M, M+1, \cdots$, involves at most an M-fold lack of closure. Clearly the same deduction for the set $\{v_m(x, 1)\}$ is possible.

We observe now that under the conditions (vii) and ($\overline{\text{vii}}$) the kernel of the normalized equation cannot be orthogonal to any continuous function.† If we assume also that

(ix) all characteristic values are simple,

then the set of solutions is biorthogonal, and since a sub-set of the characteristic functions involves only a finite lack of closure the entire set is closed.‡ Hence the sets $\{u_m(x, 1)\}$ and $\{v_m(x, 1)\}$ are closed and we have the following theorem.

THEOREM 12. *If the kernel of the normalized integral equation satisfies the conditions* (va), (vi), (vii), (viii), *and* (ix) *and the kernel of the associated equation satisfies the condition* ($\overline{\text{vii}}$), *then the sets of characteristic functions* $u_m(x)$ *and* $v_m(x)$ *are closed.*

30. **Other characteristic values and functions.** It is conceivable that the methods employed may not lead to the detection of all existent characteristic values and functions. That this is not the case, however, may be easily shown with the use of Theorem 12.§

CHAPTER 9. THE EXPANSION OF AN ARBITRARY FUNCTION

31. **The equi-convergence property.** We consider now in connection with the given integral equation the ordinary differential system (52) with solutions $y_m(x)$ and the adjoint system with solutions $z_m(x)$. The function $\phi(x, \xi, p)$ we define by the relation

$$\phi(x,\xi,p) = \sum_{m=0}^{p} [u_m(x)v_m(\xi) - y_m(x)z_m(\xi)]. \tag{56}$$

* See paper 1, §41.

† See the discussion in §10.

‡ Langer, Bulletin of the American Mathematical Society, loc. cit.

§ See paper 1, chapter 11.

Inasmuch as the functions $u_m(x, \eta)$, $v_m(x, \eta)$ become $y_m(x)$, $z_m(x)$ for $\eta=0$, and $u_m(x)$, $v_m(x)$ for $\eta=1$, we obtain from formulas (50a) for M a fixed number sufficiently large and $p>M$

$$\phi(x,\xi,p) = \phi(x,\xi,M) + \sum_{j=1}^{4} \psi_j(x,\xi,p) + \sum_{m=M+1}^{p} \frac{H(x,\xi,m)}{m^2}, \tag{56a}$$

where each of the functions ψ_i is of the form

$$\psi_i(x,\xi,p) = H(x,\xi) \sum_{m=M+1}^{p} \frac{\sin m\pi\theta_i \cos m\pi\omega_i}{m}, \tag{57}$$

and θ_i, ω_i are linear functions of x or ξ. As previously observed in the discussion of formula (55) the sums in (57) can each be written as a pair of sums of the form $\sum(\sin m\pi z)/m$. It follows from the properties of series of this type as already noted, and from formulas (56a) and (56), that

$$|\phi(x,\xi,p)| < A \qquad \text{(a constant)}, \tag{58}$$

and that for any choice of the points α and β on the interval (a, b)

$$\lim_{p\to\infty} \int_{\alpha}^{\beta} \phi(x,\xi,p)d\xi = \Phi(x),$$

the convergence being uniform and $\Phi(x)$ therefore being continuous on (a, b).

To evaluate $\Phi(x)$ we denote by $f(x)$ the function defined as follows:

$$f(x) = \begin{cases} 1 \text{ for } \alpha \leqq x \leqq \beta, \\ 0 \text{ for } a \leqq x < \alpha \text{ and } \beta < x \leqq b. \end{cases}$$

Since the system (52) is regular by (53) and has only simple characteristic values we have

$$\sum_{m=0}^{\infty} \int_{a}^{b} f(\xi)z_m(\xi)d\xi \cdot y_m(x) = \bar{f}(x),^*$$

where $\bar{f}(x)$ is bounded and differs from $f(x)$ at most in the points α, β, a, b. Except in the neighborhoods of these points moreover, the convergence is uniform as may be seen by using formulas (50a). But

$$\int_{a}^{b} f(\xi)\phi(x,\xi,p)d\xi = \int_{\alpha}^{\beta} \phi(x,\xi,p)d\xi,$$

whence

$$\Phi(x) = \sum_{m=0}^{\infty} \int_{a}^{b} f(\xi)v_m(\xi)d\xi \cdot u_m(x) - \bar{f}(x).$$

* Birkhoff, these Transactions, vol. 9 (1908), p. 390.

Multiplying this relation by $v_l(x)$ and integrating as we may because of the uniformity noted above, we obtain, since the system $\{u_l(x),\ v_l(x)\}$ is normalized and biorthogonal, the result

$$\int_a^b \Phi(x)v_l(x)dx = 0 \text{ for all } l.$$

Since the set $v_l(x)$ is closed it follows that $\Phi(x) \equiv 0$, namely that

$$\lim_{p\to\infty} \int_\alpha^\beta \phi(x,\xi,p)d\xi = 0 \tag{59}$$

uniformly in x and for all α and β on (a, b).

The relations (58) and (59) are the hypotheses of a theorem by Lebesgue* which asserts that

$$\lim_{p\to\infty} \int_a^b f(\xi)\phi(x,\xi,p)d\xi = 0$$

uniformly for any function $f(x)$ which is summable over (a, b). We have, therefore, the following theorems.

THEOREM 13. *If $f(x)$ is any function which is summable over the interval (a, b), and if*

$$F_{1p}(x) = \sum_{m=0}^{p} \int_a^b f(\xi)v_m(\xi)d\xi \cdot u_m(x),$$

$$F_{2p}(x) = \sum_{m=0}^{p} \int_a^b f(\xi)z_m(\xi)d\xi \cdot y_m(x),$$

where $u_m(x)$, $v_m(x)$ are respectively the normalized solutions of the integral equation in normal form and the associated equation and $y_m(x)$, $z_m(x)$ are respectively the normalized solutions of a related differential system (52) *and its adjoint system, then*

$$\lim_{p\to\infty} [F_{1p}(x) - F_{2p}(x)] = 0 \quad \textit{uniformly.}$$

A more explicit but less inclusive formulation may be had by drawing on the expansion theorem of the related differential system.† Thus we may state

THEOREM 13a. *If $f(x)$ is a function which on the interval (a, b) consists of a finite number of pieces, each real, continuous, and with a continuous derivative then the expansion*

* Annales de la Faculté des Sciences de Toulouse, vol. 23 ((3), vol. 1) (1909), p. 52 and p. 68.
† Birkhoff, loc. cit.

$$\sum_{m=0}^{\infty} \int_a^b f(\xi) v_m(\xi) d\xi \cdot u_m(x)$$

converges to

$$\tfrac{1}{2}[f(x+) + f(x-)] \quad \text{for } a < x < b,$$

to

$$a_1 f(a+) + a_2 f(b-) \quad \text{for } x = a,$$

and to

$$b_1 f(a+) + b_2 f(b-) \quad \text{for } x = b,$$

where a_1, a_2, b_1, b_2 are constants independent of $f(x)$.

CHAPTER 10. THE ASYMPTOTIC FORMS UNDER ALTERNATIVE HYPOTHESES

32. **The characteristic values.** The form of the characteristic equation (45a) and hence also the forms for the characteristic values and functions above were obtained as a result of the hypothesis (viii). If this hypothesis is not met, the character of the solutions is in general quite different. This will be illustrated in the following discussion.

Let us assume that the integral equation in hand is such that

$$k_{13}k_{24} - k_{14}k_{23} = 0,$$

(x)
$$C = \begin{vmatrix} k_{11} & k_{14} \\ k_{21} & k_{24} \end{vmatrix} + \begin{vmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{vmatrix} \neq 0.$$

If the forms (43a) are substituted in (45) then, certain terms in the expansion of the resulting determinant vanish because of the first of conditions (x). Substituting in the remaining terms the forms (40a) we find that the characteristic equation is of the type

(60)
$$[C + \epsilon(\lambda)]e^{\lambda(b-a)} - 2[B + \epsilon(\lambda)] + [C + \epsilon(\lambda)]e^{-\lambda(b-a)} = 0,$$

where

$$B = \begin{vmatrix} k_{13} & k_{11} \\ k_{23} & k_{21} \end{vmatrix} + \begin{vmatrix} k_{14} & k_{12} \\ k_{24} & k_{22} \end{vmatrix}.$$

If Q and $1/Q$ denote the roots of the quadratic

$$z^2 - 2\frac{B}{C}z + 1 = 0,$$

it can be shown by the method of §22 that for $|\lambda| > N$ the equation (60) admits of just one root in the neighborhood of each of the points

(61)
$$\rho_m = \frac{1}{b-a}[2m\pi i + \log Q] \quad (m = \pm M, \pm (M+1), \cdots),$$

$$\rho_{m1} = \frac{1}{b-a}[2m\pi i - \log Q].$$

We shall denote these roots respectively by λ_m and λ_{m_1}. Observing now that on the one hand $-\lambda_m$ must be a characteristic value since the equation (7) involves λ only to an even power, while on the other hand $\rho_{m_1} = -\rho_{-m}$, it becomes clear that $\lambda_{m_1} = -\lambda_{-m}$. Hence we have the following theorem:

THEOREM 14. *Under the hypothesis* (v), (vi), *and* (x), *and either* (vii) *or* (vii), *the normalized integral equation* (7) *admits of infinitely many characteristic values. For* $|m|$ *sufficiently large these values are given by the formulas*

$$\lambda_m = \frac{1}{b-a}[2m\pi i + \log Q] + \epsilon_m \quad (m = \pm M, \pm (M+1), \cdots), \tag{62}$$

$$\lambda_{m_1} = -\lambda_m.$$

33. **The characteristic functions.** It is not possible in this case to choose the constants h_1 and h_2 to satisfy (49). Instead we may choose them so that

$$h_1 k_{13} + h_2 k_{23} = 0, \quad h_1 k_{14} + h_2 k_{24} = 0. \tag{63}$$

With this choice we obtain in place of (48) the relation

$$c_1 : c_2 = -U_2(\lambda_m) : U_1(\lambda_m),$$

where

$$U_i(\lambda) = (h_1 k_{11} + h_2 k_{21}) u_i(a) + (h_1 k_{12} + h_2 k_{22}) u_i(b) - \int_a^b [h_1 \Omega_1(\xi) + h_2 \Omega_2(\xi)] u_i(\xi) d\xi.$$

Substituting in this the forms (40a) and observing that

$$e^{\lambda_m (b-a)} = Q + \epsilon(m)$$

we find

$$c_1 : c_2 = -(k_1 + \epsilon) : (k_2 + \epsilon),$$

where

$$k_1 = (h_1 k_{11} + h_2 k_{21}) + (h_1 k_{12} + h_2 k_{22}) Q^{-1},$$

$$k_2 = (h_1 k_{11} + h_2 k_{21}) + (h_1 k_{12} + h_2 k_{22}) Q.$$

Under condition (x) these two constants cannot vanish simultaneously. Inasmuch as

$$e^{\lambda_m x} = e^{\rho_m x} + \epsilon(m, x),$$

we obtain from (40a) and (42) a form for the solutions as given in the following theorem:

THEOREM 15. *Under the hypotheses* (v), (vi), (vii) *and* (x) *the characteristic functions of the normalized integral equation are given for* $|m|$ *sufficiently large by an expression of the form*

$$u_m(x) = k_1 e^{\rho_m(x-a)} - k_2 e^{-\rho_m(x-a)} + \epsilon(x,m), \tag{64}$$

the coefficients k_1 *and* k_2 *being constants, not both zero.**

34. **A further case.** It is clear that the method employed will in fact be applicable whenever the characteristic equation is of the form

$$[c_1 + \epsilon(\lambda)]e^{\lambda(b-a)} + [c_2 + \epsilon(\lambda)] + [c_3 + \epsilon(\lambda)]e^{-\lambda(b-a)} = 0, \text{ and } c_1 c_3 \neq 0.$$

The following is an example in which this holds although conditions (viii) and (x) both fail to be fulfilled.

Let $K(x, \xi)$ be any kernel satisfying condition (vi) and the further condition

$$K(x,a) \equiv K(x,b) \equiv K(a,\xi) \equiv K(b,\xi) \equiv 0. \tag{65}$$

We shall choose the parameters μ_{ij}, ν_{ij} so that $\mu_{11} = \nu_{21} = 1$, while the others all vanish. This choice is contrary to (13) but the discussion is nevertheless found to be valid. We obtain then $\boldsymbol{K}(x, \xi) \equiv K(x, \xi)$, $\theta_i(x) \equiv 0$, $\phi_i(x) \equiv 0$, $V_i(\xi) \equiv 0$, $i = 1, 2$, and $k_{11} = k_{21} = 1$, while the other quantities k_{ij} are all zero. The equations (43) reduce to $\omega_{1j} = u_j(a)$, $\omega_{2j} = u_j(b)$, and the characteristic equation is found to be of the form (45a). Hence the characteristic values are given by (45b).

In determining the characteristic functions we may choose $h_1 = 0$, $h_2 = 1$ to satisfy (63). This leads to

$$c_1 : c_2 = -(e^{-\lambda_m b} + \epsilon) : (e^{\lambda_m b} + \epsilon),$$

and with these values we find as the normalized characteristic functions

$$u_m(x) = \left(\frac{2}{b-a}\right)^{1/2} \sin \frac{m\pi(b-x)}{b-a} + \epsilon(x,m).$$

It is clear that throughout this chapter more precise forms for the characteristic values and functions are obtainable by the methods used precisely as in §23 and §26.

* If the kernel of the equation considered in paper 1 is written $K_1(x, \xi)$, the iterated kernel $K(x,\xi) = \int_a^b K_1(x,t)\, K_1(t,\xi)dt$ is readily found to satisfy the hypotheses made concerning the equation discussed in the present paper. Hence the characteristic values and solutions obtained in paper 1 must be included among those derived here. A comparison of the results shows them to be included here among the equations satisfying condition (x).

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SOME THIRD-ORDER IRREGULAR BOUNDARY VALUE PROBLEMS*

BY
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INTRODUCTION

We treat here the problem of expanding functions in infinite series whose terms are solutions of the differential equation $d^3u/dx^3+\rho^3u=0$ and various types of boundary conditions. Birkhoff† has treated a general problem of this nature for the case in which the boundary conditions are regular, and Hopkins‡ has considered a very special case of irregularity. The first part of this paper is an extension of Hopkins' work in which the distinctive feature of the boundary conditions is that two bear at only one point and the third bears at two points. In the second part of the paper is considered a very special case in which one boundary condition bears at one point, the second at another point, which is complex, and the third at both points. And in the third part is considered a case in which the boundary conditions bear symmetrically at three points, one at each point.

PART I

The boundary conditions are written $W_i(u)=0$, $i=1, 2, 3$, where $W_i(u)$ is a linear homogeneous form in $u(a)$, $u'(a)$, $u''(a)$, $u(b)$, $u'(b)$, $u''(b)$ with constant coefficients, which we shall suppose to be real. Here a and b are taken real and positive and $a<b$. By linear combination among themselves the conditions can be reduced to the equivalent set

$$\begin{aligned}
&\alpha_{12}u''(a)+\alpha_{11}u'(a)+\alpha_{10}u(a) && +\beta_{11}u'(b)+\beta_{10}u(b)=0,\\
&\alpha_{22}u''(a)+\alpha_{21}u'(a)+\alpha_{20}u(a)+\beta_{22}u''(b)+\beta_{21}u'(b)+\beta_{20}u(b)=0,\\
&\alpha_{31}u'(a)+\alpha_{30}u(a) && +\beta_{31}u'(b)+\beta_{30}u(b)=0.
\end{aligned}$$

We shall represent by $W_1(u)$, $W_2(u)$, and $W_3(u)$ the left hand sides of these three equations respectively, by $W_{ia}(u)$ that portion of $W_i(u)$ which bears at a, and by $W_{ib}(u)$ that portion which bears at b.

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* Presented to the Society, April 16, 1927; received by the editors in October, 1926.

† G. D. Birkhoff, these Transactions, vol. 9 (1908), pp. 219–231 and pp. 373–395.

‡ J. W. Hopkins, published by D. Jackson in these Transactions, vol. 20 (1919), p. 245, et seq.

The characteristic equation is $\Delta(\rho)=0$, where

$$\Delta(\rho) \equiv \begin{vmatrix} W_1(y_1) & W_1(y_2) & W_1(y_3) \\ W_2(y_1) & W_2(y_2) & W_2(y_3) \\ W_3(y_1) & W_3(y_2) & W_3(y_3) \end{vmatrix}.$$

Here y_1, y_2, and y_3 are any three linearly independent solutions of the differential equation. We shall take

$$y_i = e^{\omega_i\rho(x-a)}, \text{ where } \omega_1 = -1,\ \omega_2 = e^{\pi i/3},\ \omega_3 = e^{-\pi i/3}.$$

Inserting these functions and expanding the determinant, we find

$$\begin{aligned}\Delta(\rho) = {} & \Delta_{aaa} + \Delta_{aab}e^{\omega_3\rho(b-a)} + \Delta_{aba}e^{\omega_2\rho(b-a)} + \Delta_{baa}e^{\omega_1\rho(b-a)} \\ & + \Delta_{bbb} + \Delta_{bba}e^{\omega_3\rho(a-b)} + \Delta_{bab}e^{\omega_2\rho(a-b)} + \Delta_{abb}e^{\omega_1\rho(a-b)},\end{aligned}$$

where

$$\Delta_{aaa} = 3(\omega_2-\omega_3)\rho^3 D_a,$$

$$\Delta_{aab} = (\omega_1-\omega_2)\rho[-\omega_3 D_{a4}\rho^4 - \omega_1 D_{a3}\rho^3 - \omega_2 D_{a2}\rho^2 - \omega_3 D_{a1}\rho - \omega_1 D_{a0}],$$

$$\Delta_{aba} = (\omega_3-\omega_1)\rho[-\omega_2 D_{a4}\rho^4 - \omega_1 D_{a3}\rho^3 - \omega_3 D_{a2}\rho^2 - \omega_2 D_{a1}\rho - \omega_1 D_{a0}],$$

$$\Delta_{baa} = (\omega_2-\omega_3)\rho[D_{a4}\rho^4 + D_{a3}\rho^3 + D_{a2}\rho^2 + D_{a1}\rho + D_{a0}],$$

$$\Delta_{abb} = (\omega_2-\omega_3)\rho[D_{b4}\rho^4 + D_{b3}\rho^3 + D_{b2}\rho^2 + D_{b1}\rho + D_{b0}],$$

$$\Delta_{bab} = (\omega_3-\omega_1)\rho[-\omega_2 D_{b4}\rho^4 - \omega_1 D_{b3}\rho^3 - \omega_3 D_{b2}\rho^2 - \omega_2 D_{b1}\rho - \omega_1 D_{b0}],$$

$$\Delta_{bba} = (\omega_1-\omega_2)\rho[-\omega_3 D_{b4}\rho^4 - \omega_1 D_{b3}\rho^3 - \omega_2 D_{b2}\rho^2 - \omega_3 D_{b1}\rho - \omega_1 D_{b0}],$$

$$\Delta_{bbb} = 3(\omega_2-\omega_3)\rho^3 D_b.$$

In these expressions the D's are certain real combinations of determinants formed from the array of coefficients in the boundary conditions.

The regularity condition, which bears on the differential system, plays an important rôle in boundary value and expansion problems. In the present case the system is regular if each of the six exponentials in $\Delta(\rho)$ is present and multiplied by a polynomial of degree five in ρ. As mentioned in the introduction the regular case has been treated by Birkhoff. The cases in which the multiplying polynomials are of degrees less than five are susceptible of a treatment similar to that employed by Birkhoff, and the results are similar to those of the regular case. They will not be considered here.

We propose to treat some of the cases in which every D_a equals zero or every D_b equals zero. These we call the highly irregular cases. One of them is Hopkins' case. For the sake of definiteness we shall take $D_{bi}=0$, $i=0, 1, 2, 3, 4$.

The characteristic equation. Divide the ρ plane into six equal sectors defined by $l\pi/3 < \arg \rho < (l+1)\pi/3$, $l=0, 1, 2, 3, 4, 5$. It is only near the rays $\arg \rho = 0, 2\pi/3, 4\pi/3$ that we can hope to find characteristic numbers, for if $|\rho|$ is large and ρ is not near one of these rays in an angular sense, some one of the exponential terms in $\Delta(\rho)$ will far outbalance in numerical value all the others. But since ρ appears in the differential equation only as ρ^3, we may restrict ρ to the sector $-\pi/3 \leqq \arg \rho \leqq \pi/3$.

Consequently we need look for characteristic numbers only near the ray $\arg \rho = 0$.

We may write

$$\Delta(\rho) = e^{\omega_3 \rho(b-a)}[\phi(\rho) + \epsilon(\rho)],$$

where

$$\phi(\rho) = \Delta_{aab} + \Delta_{aba} e^{(\omega_2-\omega_1)\rho(b-a)}$$

and

$$\epsilon(\rho) = (\Delta_{aaa} + \Delta_{bbb}) e^{\omega_3 \rho(a-b)} + \Delta_{baa} e^{(\omega_1-\omega_3)\rho(b-a)}.$$

If we lay down the further restriction $-\pi/6+\delta \leqq \arg \rho \leqq \pi/3$, δ being a small positive constant, $|\epsilon(\rho)|$ will be small for $|\rho|$ large. If D_{aj} is that D_a having the largest second subscript and not being zero, we have

$$\phi(\rho) = (\omega_1 - \omega_2) \sum_{k=0}^{j} \omega_{k+1}^2 D_{ak} \rho^{k+1} + (\omega_2 - \omega_1) e^{(\omega_2-\omega_3)\rho(b-a)} \sum_{k=0}^{j} \omega_k D_{ak} \rho^{k+1}$$

$$= (\omega_1 - \omega_2)\rho^{j+1}\left[\omega_j D_{aj} + \left(\frac{1}{\rho}\right)\right][\psi(\rho) + \epsilon'(\rho)],$$

where $(1/\rho)$ means a quantity of the size of $1/\rho$ for $|\rho|$ large, $|\epsilon'(\rho)|$ is small for $|\rho|$ large, and

$$\psi(\rho) = \frac{\omega_{j+1}^2}{\omega_j} - e^{(\omega_2-\omega_3)\rho(b-a)}.$$

Here $\omega_0 = \omega_3$, $\omega_4 = \omega_1$, and $\omega_5 = \omega_2$.

We have

$$\psi(\rho) = \frac{\omega_{j+1}^2}{\omega_j} - e^{i\rho 3^{1/2}(b-a)},$$

whose zeros are given by

$$\rho = \frac{1}{i \cdot 3^{1/2}(b-a)} \log \frac{\omega_{j+1}^2}{\omega_j}.$$

This logarithm has one of the values $\pi i/3$, $-\pi i/3$, or πi, j being 0, 1, 2, 3, or 4. Hence the zeros of $\psi(\rho)$ are real and are evenly spaced along the ray $\arg \rho = 0$.

We can now discuss the zeros of $\Delta(\rho)$. For we have

$$\Delta(\rho) = (\omega_1 - \omega_2)\rho^{j+1}e^{\omega_3\rho(b-a)}\left[\omega_j D_{aj} + \left(\frac{1}{\rho}\right)\right][\psi(\rho) + \epsilon''(\rho)], \tag{1}$$

where $|\epsilon''(\rho)|$ is small for $|\rho|$ large. Draw small circles all of the same radius centered at the zeros of $\psi(\rho)$, and call S the portion of the sector $(-\pi/6 + \delta, \pi/3)$ which is exterior to these circles. As the imaginary part of ρ becomes positively infinite, $\psi(\rho) \to \omega_{j+1}^2/\omega_j$; as the imaginary part of ρ becomes negatively infinite, $\psi(\rho) \to \infty$; and $\psi(\rho)$ has a real period. Hence $|\psi(\rho)|$ has a positive minimum μ in S. Hence from (1)

$$|\Delta(\rho)| > \left|(\omega_1 - \omega_2)\rho^{j+1}e^{\omega_3\rho(b-a)}\left[\omega_j D_{aj} + \left(\frac{1}{\rho}\right)\right]\right| \cdot |\mu - |\epsilon''(\rho)||.$$

Now let $|\rho|$ be so large that $|\epsilon''(\rho)| < \mu/2$. Then we have

$$|\Delta(\rho)| > h|\rho^{j+1}e^{\omega_3\rho(b-a)}|, \tag{2}$$

and this inequality is valid for ρ in S and $|\rho|$ sufficiently large, h being independent of ρ. This inequality shows that $\Delta(\rho)$ has no zeros in S for $|\rho|$ large.

That $\Delta(\rho)$ for large $|\rho|$ does have a zero in each small circle and only one is seen from (1). As ρ travels just once around one of the circles, $\arg \psi(\rho)$ increases by just 2π, i.e., the point $\psi(\rho)$ travels just once around the origin. Hence the point $\Delta(\rho)$ will travel just once around the origin provided $|\epsilon''(\rho)|$ is small enough, and hence $\Delta(\rho)$ has just one zero in the circle. Moreover, if $|\rho|$ is large enough, the zeros of $\Delta(\rho)$ will be real. We omit the actual proof, which consists in showing by purely formal manipulations that $\overline{\Delta(\rho)} = -\Delta(\bar{\rho})$, dashes denoting conjugates. We denote the zeros of $\Delta(\rho)$ by ρ_k, $k = 1, 2, \cdots$.

The characteristic functions. A form of solution of the differential equation and the first and third boundary conditions is

$$u(x) = \frac{1}{(\omega_2 - \omega_3)\rho}\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ W_1(y_1) & W_1(y_2) & W_1(y_3) \\ W_3(y_1) & W_3(y_2) & W_3(y_3) \end{vmatrix}.$$

Substitution of ρ_k for ρ in $u(x)$ yields the kth characteristic function $u_k(x)$. We consider the characteristic functions for the case in which

$$\begin{aligned} W_1(u) &= \alpha_{12}u''(a) + \alpha_{11}u'(a) + \alpha_{10}u(a), \\ W_2(u) &= \alpha_{22}u''(a) + \alpha_{21}u'(a) + \alpha_{20}u(a) + \beta_{22}u''(b) + \beta_{21}u'(b) + \beta_{20}u(b), \\ W_3(u) &= \alpha_{31}u'(a) + \alpha_{30}u(a), \end{aligned} \tag{3}$$

the distinctive feature of which is that two of the W's bear only at one end of the interval. This is a case in which every D_k equals zero. Highly irregular cases in which at least two boundary conditions bear at both ends of the interval will be treated in a later paper.

Proceeding to develop the form of the characteristic functions, we have $u(x)=A_1(\rho)y_1(x)+A_2(\rho)y_2(x)+A_3(\rho)y_3(x)$, where the A's are certain quadratic polynomials in ρ. We may write

$$(4)\qquad u(x) = \alpha_{12}\alpha_{31}\rho^2\delta_1[\rho(x-a)] + \alpha_{12}\alpha_{30}\rho\delta_2[\rho(x-a)] + (\alpha_{11}\alpha_{30}-\alpha_{10}\alpha_{31})\delta_3[\rho(x-a)],$$

where

$$\delta_1(\rho t) = e^{\omega_1\rho t} + e^{\omega_2\rho t} + e^{\omega_3\rho t} = e^{-\rho t} + 2e^{\rho t/2}\cos\left(\frac{3^{1/2}}{2}\rho t\right),$$

$$\delta_2(\rho t) = e^{\omega_1\rho t} - \omega_3 e^{\omega_2\rho t} - \omega_2 e^{\omega_3\rho t} = e^{-\rho t} - 2e^{\rho t/2}\cos\left(-\frac{\pi}{3}+\frac{3^{1/2}}{2}\rho t\right),$$

$$\delta_3(\rho t) = e^{\omega_1\rho t} - \omega_2 e^{\omega_2\rho t} - \omega_3 e^{\omega_3\rho t} = e^{-\rho t} - 2e^{\rho t/2}\cos\left(\frac{\pi}{3}+\frac{3^{1/2}}{2}\rho t\right).$$

These δ functions are seen to be real if ρ and t are real, and they satisfy the relations

$$\frac{d}{dt}\delta_1(\rho t) = -\rho\delta_3(\rho t), \quad \frac{d}{dt}\delta_2(\rho t) = -\rho\delta_1(\rho t), \quad \frac{d}{dt}\delta_3(\rho t) = -\rho\delta_2(\rho t);$$

$$\delta_1(0) = 3, \quad \delta_2(0) = 0, \quad \delta_3(0) = 0, \text{ etc.}$$

Hence $u(x)$ is real if both ρ and x are real. We may write

$$(5)\qquad \begin{aligned} u(x) &= e^{-\rho(x-a)}[\alpha_{12}\alpha_{31}\rho^2 + \alpha_{12}\alpha_{30}\rho + (\alpha_{11}\alpha_{30}-\alpha_{10}\alpha_{31})] \\ &+ 2e^{\rho(x-a)/2}\left[\alpha_{12}\alpha_{31}\rho^2\cos\left\{\frac{3^{1/2}}{2}\rho(x-a)\right\} - \alpha_{12}\alpha_{30}\rho\cos\left\{-\frac{\pi}{3}\right.\right. \\ &\left.\left.+\frac{3^{1/2}}{2}\rho(x-a)\right\} - (\alpha_{11}\alpha_{30}-\alpha_{10}\alpha_{31})\cos\left\{\frac{\pi}{3}+\frac{3^{1/2}}{2}\rho(x-a)\right\}\right]. \end{aligned}$$

Necessary conditions for convergence of formal series. We state the following

LEMMA. *If the series*

$$(6)\qquad \sum_{k=1}^{\infty} a_k u_k(x)$$

converges uniformly in some interval $\alpha \leqq x \leqq \beta$, *where* $0 < \alpha < \beta < \pi$, *then* $|a_k| < \mu e^{-\rho_k(x_0 - a)/2}$, *where* x_0 *is any constant less than* β, *and* μ *is independent of* k.

Noting that at least one of the cosine terms in (5) must be present, for otherwise we should not have more than two boundary conditions (3), the proof of this lemma is the same as that of the corresponding one in Hopkins' paper.

In the series (6) we now allow x to be complex. Writing $x - a = t$, we have

$$|\delta_i[\rho_k(x-a)]| < \sum_{j=1}^{3} |e^{\omega_j \rho_k t}| ;$$

and, calling $t = \xi + \eta i$, ξ and η both being real,

$$|\delta_i[\rho_k(x-a)]| < e^{-\rho_k \xi} + e^{\rho_k(\xi - \eta \cdot 3^{1/2})/2} + e^{\rho_k(\xi + \eta \cdot 3^{1/2})/2}.$$

Here k is supposed to be so large that ρ_k is known to be real. Let $t_0 = x_0 - a$, and choose a number x_1 less than x_0 but greater than a. Let $t_1 = x_1 - a$. Then $t_1 - t_0 = x_1 - x_0 < 0$. We shall have

$$\left.\begin{aligned} e^{-\rho_k \xi} &< e^{\rho_k t_1/2} \\ e^{\rho_k(\xi - \eta \cdot 3^{1/2})/2} &< e^{\rho_k t_1/2} \\ e^{\rho_k(\xi + \eta \cdot 3^{1/2})/2} &< e^{\rho_k t_1/2} \end{aligned}\right\} \text{ provided } \begin{cases} \xi > -\tfrac{1}{2} t_1 \\ \xi - \eta \cdot 3^{1/2} < t_1 \\ \xi + \eta \cdot 3^{1/2} < t_1. \end{cases}$$

All of these inequalities are satisfied if t is within the triangle whose sides are $\xi = -t_1/2$, $\xi - \eta \cdot 3^{1/2} = t_1$, $\xi + \eta \cdot 3^{1/2} = t_1$, i.e., if x is within the equilateral triangle centered at $x = a$ and having one vertex at $x = x_1$. Hence we have within this triangle

$$|u_k(x)| < 3e^{\rho_k(x_1 - a)/2} \quad \text{and} \quad |a_k u_k(x)| < \frac{3g}{h} e^{\rho_k(x_1 - x_0)/2}.$$

But the expression on the right is the general term of a convergent series of positive constants. Thus we have proved

THEOREM 1. *Under the hypothesis of the above lemma series* (6) *converges uniformly in the interior of the equilateral triangle centered at* $x = a$ *and having one vertex at* $x = x_0$, *and represents therein an analytic function of the complex variable* x.

This theorem presents a restriction on the type of function representable by series (6). But there are further restrictions. We may write $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k = \sum_{k=1}^{\infty} a_k u_k(x)$, where $c_k = (1/k!) f^{(k)}(a)$. By the first of

(3) we have $\alpha_{12}u_k''(a)+\alpha_{11}u_k'(a)+\alpha_{10}u_k(a)=0$. Use of the differential equation satisfied by $u_k(x)$ gives

$$\alpha_{12}u_k^{(2+3n)}(a) + \alpha_{11}u_k^{(1+3n)}(a) + \alpha_{10}u_k^{(3n)}(a) = 0 \ (n = 0,1,2, \cdots).$$

Hence

$$\alpha_{12}f^{(2+3n)}(a) + \alpha_{11}f^{(1+3n)}(a) + \alpha_{10}f^{(3n)}(a) = 0 \quad (n = 0,1,2, \cdots).$$

Hence

$$\alpha_{12}(2 + 3n)(1 + 3n)c_{2+3n} + \alpha_{11}(1 + 3n)c_{1+3n} + \alpha_{10}c_{3n} = 0 \tag{7}$$
$$(n = 0,1,2, \cdots).$$

Now write $f(x)=\phi_1(x)+\phi_2(x)+\phi_3(x)$, where

$$\phi_1(x) = \sum_{n=0}^{\infty} c_{3n}(x - a)^{3n}, \qquad \phi_2(x) = \sum_{n=0}^{\infty} c_{1+3n}(x - a)^{1+3n},$$

$$\phi_3(x) = \sum_{n=0}^{\infty} c_{2+3n}(x - a)^{2+3n},$$

and form the expression

$$\begin{aligned}\alpha_{12}\phi_3''(x) + \alpha_{11}\phi_2'(x) + \alpha_{10}\phi_1(x) &= \alpha_{12}\sum_{n=0}^{\infty} c_{2+3n}(2 + 3n)(1 + 3n)(x - a)^{3n} \\ &\quad + \alpha_{11}\sum_{n=0}^{\infty} c_{1+3n}(1 + 3n)(x - a)^{3n} \\ &\quad + \alpha_{10}\sum_{n=0}^{\infty} c_{3n}(x - a)^{3n} \\ &= \sum_{n=0}^{\infty} [\alpha_{12}(2 + 3n)(1 + 3n)c_{2+3n} + \alpha_{11}(1 + 3n)c_{1+3n} + \alpha_{10}c_{3n}](x - a)^{3n}.\end{aligned}$$

By (7) the coefficient of every power of $(x-a)$ in this series is seen to equal zero. Hence

$$\alpha_{12}\phi_3''(x) + \alpha_{11}\phi_2'(x) + \alpha_{10}\phi_1(x) \equiv 0. \tag{8'}$$

Similarly

$$\alpha_{31}\phi_2'(x) + \alpha_{30}\phi_1(x) \equiv 0. \tag{8''}$$

Convergence proof. The conditions of analyticity at $x=a$ and identities (8) together with certain auxiliary conditions imposed on $f(x)$ for the purpose of accomplishing readily the convergence proof constitute sufficient conditions that $f(x)$ be expansible in a series of type (6). The convergence proof will now be given, and the theorem stated after the proof is finished.

We introduce a Green's function and a contour integral equal to the sum $I_n(x)$ of the first n terms of the formal series arising from $f(x)$.* Let

$$g(x,s,\rho) = \pm\frac{1}{2}\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(s) & y_2''(s) & y_3'(s) \\ y_1(s) & y_2(s) & y_3(s) \end{vmatrix} \div \begin{vmatrix} y_1''(s) & y_2''(s) & y_3''(s) \\ y_1'(s) & y_2'(s) & y_3'(s) \\ y_1(s) & y_2(s) & y_3(s) \end{vmatrix}$$

$$= \pm\frac{1}{2}\sum_1^3 y_i(x)z_i(s) = \pm\frac{1}{6\rho^2}\delta_3[\rho(x-s)], \qquad \begin{cases} + \text{ if } x > s \\ - \text{ if } x < s. \end{cases}$$

Here the definitions of the functions $z_1(s)$ are obvious, and the reduction to the δ function is straightforward. It turns out that

$$z_i(s) = -\frac{\omega_i}{3\rho^2}e^{\omega_i\rho(a-s)}.$$

Let

$$G(x,s,\rho) = -\frac{1}{\Delta(\rho)}\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & g(x,s,\rho) \\ W_1(y_1) & W_1(y_2) & W_1(y_3) & W_1(g) \\ W_2(y_1) & W_2(y_2) & W_2(y_3) & W_2(g) \\ W_3(y_1) & W_3(y_2) & W_3(y_3) & W_3(g) \end{vmatrix}.$$

Then

$$I_n(x) = \frac{1}{2\pi i}\int_{\gamma_n}\int_a^b 3\rho^2 G(x,s,\rho)f(s)\,ds\,d\rho,$$

where γ_n is an arc of a circle centered at $\rho=0$ and extending from arg $\rho=-\pi/3$ to arg $\rho=\pi/3$. It is supposed that the radius of γ_n is greater than $|\rho_n|$ but less than $|\rho_{n+1}|$.

Before considering the integration we alter the form of G. Multiplying the first, second, and third columns of the determinant in G by $\frac{1}{2}z_1(s)$, $\frac{1}{2}z_2(s)$, and $\frac{1}{2}z_3(s)$ respectively, and adding to the fourth column, we get

$$G = -\frac{1}{\Delta(\rho)}\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & g(x,s,\rho) + \frac{1}{2}\sum_{i=1}^3 y_i(x)z_i(s) \\ W_1(y_1) & W_1(y_2) & W_1(y_3) & 0 \\ W_2(y_1) & W_2(y_2) & W_2(y_3) & 2W_{2b}(g) \\ W_3(y_1) & W_3(y_2) & W_3(y_3) & 0 \end{vmatrix},$$

which on being expanded gives

$$G = g + \frac{1}{2}\sum_1^3 y_i(x)z_i(s) + \frac{2(\omega_2-\omega_3)\rho W_{2b}(g)}{\Delta(\rho)}u(x).$$

* See Birkhoff, loc. cit., p. 379 and p. 390.

Further, $\Delta(\rho)=(\omega_2-\omega_3)\rho[3\rho^2D_a-W_{2b}(u)]$, so that

$$G=\frac{1}{3\rho^2}\delta_3[\rho(x-s)]+\frac{2W_{2b}(g)u(x)}{3\rho^2D_a-W_{2b}(u)} \qquad \text{if } x>s,$$

$$G=\frac{2W_{2b}(g)u(x)}{3\rho^2D_a-W_{2b}(u)} \qquad \text{if } x<s.$$

Thus we obtain the form of contour integral with which it is convenient to work, namely,

$$(9)\qquad I_n(x)=\frac{1}{2\pi i}\int_{\gamma_n}\int_a^x \delta_3[\rho(x-s)]f(s)ds\,d\rho + \frac{1}{2\pi i}\int_{\gamma_n}\int_a^b 3\rho^2\frac{2W_{2b}(g)u(x)}{3\rho^2D_a-W_{2b}(u)}f(s)ds\,d\rho.$$

The following six functions are introduced:

$$P_1(x)=\rho^2\phi_1(a)\delta_2[\rho(x-a)]-\rho\phi_2'(a)\delta_3[\rho(x-a)]+\phi_3''(a)\delta_1[\rho(x-a)],$$
$$P_2(x)=\rho^2\phi_1(a)\delta_3[\rho(x-a)]-\rho\phi_2'(a)\delta_1[\rho(x-a)]+\phi_3''(a)\delta_2[\rho(x-a)],$$
$$P_3(x)=\rho^2\phi_1(a)\delta_1[\rho(x-a)]-\rho\phi_2'(a)\delta_2[\rho(x-a)]+\phi_3''(a)\delta_3[\rho(x-a)],$$

$$R_1(x)=-\rho\int_a^x\delta_3[\rho(x-s)]\phi_1''(s)ds+\int_a^x\delta_1[\rho(x-s)]\phi_2'''(s)ds - \frac{1}{\rho}\int_a^x\delta_2[\rho(x-s)]\phi_3^{iv}(s)ds,$$

$$R_2(x)=-\rho\int_a^x\delta_1[\rho(x-s)]\phi_1''(s)ds+\int_a^x\delta_2[\rho(x-s)]\phi_2'''(s)ds - \frac{1}{\rho}\int_a^x\delta_3[\rho(x-s)]\phi_3^{iv}(s)ds,$$

$$R_3(x)=-\rho\int_a^x\delta_2[\rho(x-s)]\phi_1''(s)ds+\int_a^x\delta_3[\rho(x-s)]\phi_2'''(s)ds - \frac{1}{\rho}\int_a^x\delta_1[\rho(x-s)]\phi_3^{iv}(s)ds.$$

Using $f(x)=\phi_1(x)+\phi_2(x)+\phi_3(x)$, and integrating by parts the term containing $\phi_1(s)$ twice, that containing $\phi_2(s)$ three times, and that containing $\phi_3(s)$ four times, we find that

$$\int_a^x\delta_3[\rho(x-s)]f(s)ds=\frac{3}{\rho}f(x)-\frac{3}{\rho^4}\phi_3'(x)-\frac{1}{\rho^3}\{P_3(x)+R_3(x)\},$$

and

$$3\rho^2\int_a^b 2W_{2b}(g)f(s)ds = \beta_{22}\left[\frac{3}{\rho}\{\phi_2''(b)+\phi_3''(b)\}-\frac{1}{\rho}\{P_1(b)+R_1(b)\}\right]$$
$$+\beta_{21}\left[\frac{3}{\rho}f'(b)+\frac{1}{\rho^2}\{P_2(b)+R_2(b)\}\right]$$
$$+\beta_{20}\left[\frac{3}{\rho}\left\{f(b)-\frac{\phi_3'''(b)}{\rho^3}\right\}-\frac{1}{\rho^3}\{P_3(b)+R_3(b)\}\right].$$

These results are to be substituted in (9), and the terms separated into three groups, one containing $f(x)$, the second containing $\phi_3'''(x)$ and terms involving the values of the ϕ's and their derivatives at $x=b$, and the third containing terms involving the P's and R's.

The ρ-integration is taken in two steps, one being an integration over γ_n', which is that part of γ_n on which $\arg\rho>0$, and the other an integration over γ_n'', on which $\arg\rho<0$. We consider first the integration over γ_n'.

Since the inequality $|u(x)|<l|\rho^2e^{\omega_3\rho(x-a)}|$, where l is independent of ρ, and inequality (2) are both valid on γ_n', we have

$$(10)\qquad \left|\frac{u(x)}{3\rho^2D_a-W_{2b}(u)}\right|<l\cdot 3^{1/2}\left|\frac{\rho^3e^{\omega_3\rho(x-a)}}{\Delta(\rho)}\right|<\frac{l\cdot 3^{1/2}}{h}|\rho^{2-j}e^{\omega_3\rho(x-b)}|.$$

The exponential in the last expression has an exponent whose real part is negative over the entire arc γ_n'. For this reason the second group of terms arising from (9) yields in the ρ-integration over γ_n' only terms which tend uniformly to zero as $n\to\infty$, x being restricted to a closed interval interior to (a, b) not containing either a or b. We shall refer to terms arising from (9) which tend to zero in this manner as terms which can be neglected.

The terms of group three are all put over the denominator $3\rho^2D_a-W_{2b}(u)$, and it is then seen that that part of the numerator which contains D_a leads to terms which can be neglected. The terms of the numerator containing P's are

$$\frac{1}{\rho^3}\left[\beta_{22}\{P_3(x)u''(b)-\rho^2P_1(b)u(x)\}+\beta_{21}\{P_3(x)u'(b)+\rho P_2(b)u(x)\}\right.$$
$$\left.+\beta_{20}\{P_3(x)u(b)-P_3(b)u(x)\}\right].$$

If in this expression we replace each P and each u by its value in terms of the δ functions, the coefficients of the β's are each seen to equal zero. Consequently we have

$$(11)\quad I_n'(x) = \frac{1}{2}f(x) + \epsilon_n(x) - \frac{1}{2\pi i}\int_{\gamma_n'}\left\{\frac{\beta_{22}[\rho^2 u(x)R_1(b) - u''(b)R_3(x)] - \beta_{21}[\rho u(x)R_2(b) + u'(b)R_3(x)]}{\rho^3[3\rho^2 D_a - W_{2b}(u)]} + \frac{\beta_{20}[u(x)R_3(b) - u(b)R_3(x)]}{\rho^3[3\rho^2 D_a - W_{2b}(u)]}\right\}d\rho,$$

in which $\epsilon_n(x)$ stands for that portion of $I_n'(x)$ arising from the neglected terms.

The coefficient of each β in the numerator of this integrand is to be considered by itself. We define variable y by means of the equation

$$y - a = (x - a)e^{i(\pi - \arg \rho)} \quad .$$

In the following we also change the variables of integration by the transformations $s = a + \omega_i^2(t - a)$. We have

$$\begin{aligned}\int_a^x \delta_2[\rho(x - s)]\phi_1''(s)ds &= \sum_1^3 \omega_i^2 e^{\omega_i \rho x}\int_a^x e^{-\omega_i \rho s}\phi_1''(s)ds \\ &= \sum_1^3 e^{\omega_i \rho(x-a)}\int_a^{a-\omega_i(x-a)} e^{\rho(t-a)}\phi_1''(t)dt \\ &= \delta_1[\rho(x - a)]\mu_1 + \sum_1^3 e^{\omega_i \rho(x-a)}\nu_{i1},\end{aligned}$$

where

$$\mu_j = \int_a^y e^{\rho(t-a)}\phi_j^{(j+1)}(t)dt$$

and

$$\nu_{ij} = \int_y^{a-\omega_i(x-a)} e^{\rho(t-a)}\phi_j^{(j+1)}(t)dt.$$

In order to render these steps and the following ones valid we require each of the functions $\phi_j(t)$ to be analytic in a circle centered at $t = a$. Let $r < b - a$ be the radius of this circle. We require the real number x to be within this circle. Then the complex variable y, as well as the numbers $a - \omega_i(x - a)$ will be within the circle. Hence, taking the paths of t-integration as straight lines, these paths will lie entirely within the circle. By steps similar to those above we find

$$\int_a^x \delta_3[\rho(x - s)]\phi_2'''(s)ds = \delta_2[\rho(x - a)]\mu_2 + \sum_1^3 \omega_i^3 \nu_{i2} e^{\omega_i \rho(x-a)},$$

and

$$\int_a^x \delta_1[\rho(x-s)]\phi_3^{iv}(s)ds = \delta_3[\rho(x-a)]\mu_3 - \sum_1^3 \omega_i\nu_{i3}e^{\omega_i\rho(x-a)}.$$

From these we get

$$(12)\quad R_3(x) = -\rho\delta_1[\rho(x-a)]\mu_1 + \delta_2[\rho(x-a)]\mu_2 - \frac{1}{\rho}\delta_3[\rho(x-a)]\mu_3 + A_3,$$

where

$$(13)\quad A_3 = \sum_1^3 e^{\omega_i\rho(x-a)}\left[-\rho\nu_{i1} + \omega_i^2\nu_{i2} + \frac{1}{\rho}\omega_i\nu_{i3}\right].$$

The reason for breaking up $R_3(x)$ in this way is that the portion of $R_3(x)$ containing the μ terms does not yield in the ρ-integration a quantity which has zero for its limit as $n\to\infty$, but does cancel against a similar portion of the other part of the coefficient of the β under consideration. The A_3 portion of $R_3(x)$ does yield in the ρ-integration a quantity which has zero for its limit as $n\to\infty$.

We have also

$$(14)\quad R_3(b) = -\rho\int_a^x \delta_2[\rho(b-s)]\phi_1''(s)ds + \int_a^x \delta_3[\rho(b-s)]\phi_2'''(s)ds - \frac{1}{\rho}\int_a^x \delta_1[\rho(b-s)]\phi_3^{iv}(s)ds + C_3,$$

where

$$C_3 = -\rho\int_x^b \delta_2[\rho(b-s)]\phi_1''(s)ds + \int_x^b \delta_3[\rho(b-s)]\phi_2'''(s)ds - \frac{1}{\rho}\int_x^b \delta_1[\rho(b-s)]\phi_3^{iv}(s)ds.$$

But

$$\int_a^x \delta_2[\rho(b-s)]\phi_1''(s)ds = \delta_1[\rho(b-a)]\mu_1 + \sum_1^3 \nu_{i1}e^{\omega_i\rho(b-a)},$$

$$(15)\quad \int_a^x \delta_3[\rho(b-s)]\phi_2'''(s)ds = \delta_2[\rho(b-a)]\mu_2 + \sum_1^3 \omega_i^2\nu_{i2}e^{\omega_i\rho(b-a)},$$

$$\int_a^x \delta_1[\rho(b-s)]\,\phi^{iv}(s)ds = \delta_3[\rho(b-a)]\mu_3 - \sum_1^3 \omega_i\nu_{i3}e^{\omega_i\rho(b-a)}.$$

Hence (14) becomes

$$(16)\quad R_3(b) = -\rho\delta_1[\rho(b-a)]\mu_1 + \delta_2[\rho(b-a)]\mu_2 - \frac{1}{\rho}\delta_3[\rho(b-a)]\mu_3 + B_3 + C_3,$$

where

$$(17)\qquad B_3 = \sum_1^3 e^{\omega_i\rho(b-a)}\left[-\rho\nu_{i1} + \omega_i^2\nu_{i2} + \frac{1}{\rho}\omega_i\nu_{i3}\right].$$

From (4), (12), and (16) we get for the coefficient of β_{20} in the numerator of the integrand in (11) the expression

$$(18)\qquad \begin{aligned}&\rho^2\{\delta_1[\rho(x-a)]\delta_2[\rho(b-a)] - \delta_2[\rho(x-a)]\delta_1[\rho(b-a)]\}\{\alpha_{12}\alpha_{31}\mu_2\\ &+ \alpha_{12}\alpha_{30}\mu_1\} + \rho\{\delta_3[\rho(x-a)]\delta_1[\rho(b-a)] - \delta_1[\rho(x-a)]\delta_3[\rho(b-a)]\}\\ &\cdot\{\alpha_{12}\alpha_{31}\mu_3 - (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\mu_1\} + \{\delta_3[\rho(x-a)]\delta_2[\rho(b-a)]\\ &- \delta_2[\rho(x-a)]\delta_3[\rho(b-a)]\}\{\alpha_{12}\alpha_{30}\mu_3 + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\mu_2\}\\ &+ u(x)(B_3 + C_3) - u(b)A_3.\end{aligned}$$

Now

$$\alpha_{31}\mu_2 + \alpha_{30}\mu_1 = \int_a^v e^{\rho(t-a)}[\alpha_{31}\phi_2'''(t) + \alpha_{30}\phi_1''(t)]dt,$$

which is seen to equal zero provided (8″) is satisfied. For similar reasons $\alpha_{12}\alpha_{31}\mu_3 - (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\mu_1 = 0$ and $\alpha_{12}\alpha_{30}\mu_3 + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\mu_2 = 0$. Hence (18) reduces to $u(x)(B_3+C_3) - u(b)A_3$.

In exactly the same way the coefficients of β_{21} and β_{22} in the numerator of the integrand in (11) reduce, and we obtain

$$(19)\qquad I_n'(x) = \tfrac{1}{2}f(x) + \epsilon_n(x) - \frac{1}{2\pi i}\int_{\gamma_n'}\left\{\frac{\beta_{22}[\rho^2 u(x)(B_1+C_1) - u''(b)A_3]}{\rho^2[3\rho^2 D_a - W_{2b}(u)]}\right.$$
$$\left. - \frac{\beta_{21}[\rho u(x)(B_2+C_2) + u'(b)A_3] + \beta_{20}[u(x)(B_3+C_3) - u(b)A_3}{\rho^3[3\rho^2 D_a - W_{2b}(u)]}\right\}d\rho,$$

where

$$B_2 = \sum_1^3 e^{\omega_i\rho(b-a)}\left[\rho\omega_i\nu_{i1} + \nu_{i2} - \frac{1}{\rho}\omega_i^2\nu_{i3}\right],$$

$$B_1 = \sum_1^3 e^{\omega_i\rho(b-a)}\left[-\rho\omega_i^2\nu_{i1} - \omega_i\nu_{i2} - \frac{1}{\rho}\nu_{i3}\right],$$

$$C_2 = -\rho\int_x^b \delta_1[\rho(b-s)]\phi_1''(s)ds + \int_x^b \delta_2[\rho(b-s)]\phi_2'''(s)ds$$
$$-\frac{1}{\rho}\int_x^b \delta_3[\rho(b-s)]\phi_3^{iv}(s)ds,$$

$$C_1 = -\rho\int_x^b \delta_3[\rho(b-s)]\phi_1''(s)ds + \int_x^b \delta_1[\rho(b-s)]\phi_2'''(s)ds$$
$$-\frac{1}{\rho}\int_x^b \delta_2[\rho(b-s)]\phi_3^{iv}(s)ds.$$

Consider next the portions of the integral in (19) arising from the terms containing A_3. We see from (13) that A_3 itself contains nine integrals of the type

$$\int_y^{a-\omega_i(x-a)} e^{\rho(t-a)}\phi_j^{(j+1)}(t)dt.$$

In every one of these integrals $|e^{\rho(t-a)}|$ takes on its largest value along the line of integration at the upper limit. For, as t runs from y to $a-\omega_i(x-a)$, $\rho(t-a)$ runs from $-|\rho|(x-a)$ to $-\omega_i\rho(x-a)$. The points in the complex plane representing this latter pair of numbers are at equal distances from the origin and the former is on the negative axis of reals. Consequently the real part of $\rho(t-a)$ increases steadily as t runs from the lower limit of integration to the upper. Hence, if $|\phi_j^{(j+1)}(t)|<M$, a constant, for all j entering in this discussion, we have $|\nu_{ij}|<2M(b-a)|e^{-\omega_i\rho(x-a)}|$. Hence $|A_3|<K|\rho|$, where K is independent of ρ and of x. The portion of the integrand containing A_3 is

$$\frac{W_{2b}(u)}{3\rho^2D_a - W_{2b}(u)} \cdot \frac{A_3}{\rho^3}.$$

We have seen that the first of these factors is a bounded function of ρ for $|\rho|$ large and on γ_n'. Hence the portion of the integrand containing A_3 can be neglected.

Consider now the remaining portion of the numerator of the integrand in (19). Formal reductions yield

$$\begin{aligned} &u(x)[\beta_{22}\rho^2B_1 - \beta_{21}\rho B_2 + \beta_{20}B_3] \\ &= \sum_{i=1}^{3} e^{\omega_i\rho(b-a)}\{\omega_i^2D_{a4}\rho^5 - \omega_iD_{a3}\rho^4 + D_{a2}\rho^3 + \omega_i^2D_{a1}\rho^2 \\ &\quad - \omega_iD_{a0}\rho\}\{\delta_1[\rho(x-a)]\nu_{i1} - \frac{1}{\rho}\delta_2[\rho(x-a)]\nu_{i2} + \frac{1}{\rho^2}\delta_3[\rho(x-a)]\nu_{i3}\}, \end{aligned}$$

and show that $u(x)[\beta_{22}\rho^2C_1 - \beta_{21}\rho C_2 + \beta_{20}C_3]$ equals a sum of five products of which the ith is of the form

$$\begin{aligned} D_{ai}\rho^{i+1}\Big\{&\delta_1[\rho(x-a)]\int_x^b \delta_j[\rho(b-s)]\phi_1''(s)ds \\ &- \frac{1}{\rho}\delta_2[\rho(x-a)]\int_x^b \delta_k[\rho(b-s)]\phi_2'''(s)ds \\ &+ \frac{1}{\rho^2}\delta_3[\rho(x-a)]\int_x^b \delta_l[\rho(b-s)]\phi_3^{iv}(s)ds\Big\}, \end{aligned}$$

j, k, l having the values 1, 2, 3 in various orders in these terms. Hence

$$\begin{aligned}
&| u(x)[\beta_{22}\rho^2 B_1 - \beta_{21}\rho B_2 + \beta_{20}B_3]| \\
&< 6M(b-a)|\, e^{\omega_3\rho(x-a)}|\, \sum_{i=1}^{3} |\, e^{\omega_i\rho(b-a)}|\cdot|\, D_{ij}\rho^{j+1}|\cdot|\, e^{-\omega_i\rho(x-a)}| \\
&< K|\, \rho^{j+1}e^{\omega_j\rho(b-a)}|\,,
\end{aligned}$$

where K is independent of ρ and of x. Also

$$| u(x)[\beta_{22}\rho^2 C_1 - \beta_{21}\rho C_2 + \beta_{20}C_3]| < K'|\, \rho^{j+1}e^{\omega_j\rho(b-a)}|\,,$$

K' being independent of ρ and of x. Hence the portion of the integrand in (19) containing the B and C terms is less in numerical value than some constant divided by $|\rho^2|$, and can be neglected. This leaves $I_n'(x) = \frac{1}{2}f(x) + \epsilon_n(x)$, where $\epsilon_n(x)$ tends uniformly to zero as a limit when $n \to \infty$.

The portion of $I_n(x)$ arising from integration over the arc γ_n'' is treated in a fashion quite similar to the above treatment of $I_n'(x)$ and with similar results. We omit the consideration of it, and state immediately

THEOREM 2. *If the boundary conditions can be written in the form* (3) *with at least one D_a not zero; if $f(x) = \phi_1(x) + \phi_2(x) + \phi_3(x)$, where $\phi_1(x) = \sum_0^\infty c_{3n}\cdot(x-a)^{3n}$, $\phi_2(x) = \sum_0^\infty c_{1+3n}(x-a)^{1+3n}$, $\phi_3(x) = \sum_0^\infty c_{2+3n}(x-a)^{2+3n}$ are power series convergent in and on the boundary of a circle centered at $x=a$ and of radius $r<b-a$ which satisfy the identities $\alpha_{12}\phi_3''(x) + \alpha_{11}\phi_2'(x) + \alpha_{10}\phi_1(x) \equiv 0$, $\alpha_{31}\phi_2'(x) + \alpha_{30}\phi_1(x) \equiv 0$; and if also $\phi_1''(x)$, $\phi_2'''(x)$, $\phi_3^{iv}(x)$ are continuous for real values of x from a to b, then the formal series for $f(x)$ of type* (6) *converges uniformly to $f(x)$ along the segment of the real axis from a to $a+r$.*

From Theorem 1 it follows as a corollary to Theorem 2 that the formal series will converge uniformly to $f(x)$ in the equilateral triangle whose center is at $x=a$ and one vertex of which is at $x=a+r$.

We come next to the case in which two of the boundary conditions bear only at $x=b$. We do no more than state the results. Let the boundary conditions be written in the form

$$\begin{aligned}
\beta_{12}u''(b) + \beta_{11}u'(b) + \beta_{10}u(b) &= 0, \\
\alpha_{22}u''(a) + \alpha_{21}u'(a) + \alpha_{20}u(a) + \beta_{22}u''(b) + \beta_{21}u'(b) + \beta_{20}u(b) &= 0, \\
\beta_{23}u'(b) + \beta_{30}u(b) &= 0.
\end{aligned}$$

The characteristic numbers are found to lie along the rays $\arg \rho = \pi/3$, π, $5\pi/3$. We restrict ρ to the sector $2\pi/3 \leqq \arg \rho \leqq 4\pi/3$, using the characteristic numbers along the ray $\arg \rho = \pi$.

On the assumption of uniform convergence of $\sum_1^\infty b_k u_k(x)$ in an interval $\alpha \leqq x \leqq \beta$, where $a<\alpha<\beta<b$, we obtain a restriction on the b_k's which is of sufficient strength to allow us to show that the series converges uniformly in the equilateral triangle centered at $x=b$ and having one vertex at $x=\alpha$. A formal property of the characteristic functions then shows that the series necessarily represents a function of the form $f(x)=\phi_1(x)+\phi_2(x)+\phi_3(x)$, where

$$\phi_1(x) = \sum_0^\infty c_{3n}(x-b)^{3n}, \qquad \phi_2(x) = \sum_0^\infty c_{1+3n}(x-b)^{1+3n},$$

$$\phi_3(x) = \sum_0^\infty c_{2+3n}(x-b)^{2+3n},$$

each ϕ being analytic in the triangle, and that the ϕ's must satisfy the identities

$$\beta_{12}\phi_3''(x) + \beta_{11}\phi_2'(x) + \beta_{10}\phi_1(x) \equiv 0, \quad \beta_{31}\phi_2'(x) + \beta_{30}\phi_1(x) \equiv 0.$$

If we now take a function $f(x)$ satisfying the above conditions and in addition such that $\phi_1''(x), \phi_2'''(x)$, and $\phi_3^{iv}(x)$ are continuous from a to b, the formal series for $f(x)$ of type $\sum_1^\infty b_k u_k(x)$ will converge uniformly to $f(x)$ in the interval $(b-r, b)$, r being the radius of the circle in which the ϕ's are assumed to be analytic.

PART II

We have made reference to a very special case treated by Hopkins in which the boundary conditions can be written $u(0)=u'(0)=u(\pi)=0$. The characteristic functions in this case are $u_k(x)=\delta_3(\rho_k x)$, $k=1, 2, \cdots$, and they satisfy the functional equations $u_k(-\omega_2 x)=-\omega_3 u_k(x)$, $u_k(-\omega_3 x) = -\omega_2 u_k(x)$. Hence $u_k(-\omega_2\pi)=u_k(-\omega_3\pi)=0$. Also, denoting differentiations with respect to x by accents, $u_k''(-\omega_3 x)=u_k''(x)$. Hence $u_k''(-\omega_3\pi)=0$. From these we select

$$u(\pi) = u(-\omega_3\pi) = 0, \quad u''(\pi) = u''(-\omega_3\pi), \tag{20}$$

which we shall take as boundary conditions. They will be seen to lead to a larger class of characteristic functions than Hopkins', but one which includes his.

Use of the first two of (20) gives

$$u(x) = (e^{2\omega_3\rho\pi} - e^{-\omega_3\rho\pi})e^{\omega_1\rho x} + (e^{2\omega_1\rho\pi} - e^{-\omega_1\rho\pi})e^{\omega_3\rho x} + (e^{2\omega_1\rho\pi} - e^{-\omega_1\rho\pi})e^{\omega_1\rho x}, \tag{21}$$

at least to a factor independent of x. The characteristic equation is

$$\Delta(\rho) \equiv \begin{vmatrix} e^{\omega_1\rho\pi} & e^{\omega_2\rho\pi} & e^{\omega_3\rho\pi} \\ e^{\omega_1\rho\pi} & e^{\omega_2\rho\pi} & e^{\omega_3\rho\pi} \\ \omega_1^2\rho^2(e^{\omega_1\rho\pi} - e^{\omega_3\rho\pi}) & \omega_2^2\rho^2(e^{\omega_2\rho\pi} - e^{\omega_1\rho\pi}) & \omega_3^2\rho^2(e^{\omega_3\rho\pi} - e^{\omega_2\rho\pi}) \end{vmatrix}$$

$$\equiv (\omega_3 - \omega_1)\rho^2\delta_3(\rho\pi)\delta_3(-\rho\pi) = 0. \tag{22}$$

The fact that the characteristic equation factors in the particular way indicated here is what suggested the choice of boundary conditions under consideration. This choice, of course, is only one of many that might be made. The factor $\delta_3(\rho\pi)$ of (22) has for its zeros precisely Hopkins' characteristic numbers, which we denote by ρ_{1k}. The other factor $\delta_3(-\rho\pi)$ has the negatives of the numbers ρ_{1k} for its zeros; we denote these by ρ_{2k}, and write $\rho_{2k} = -\rho_{1k}$.

We have

$$e^{\omega_1\rho_{1k}\pi} - \omega_2 e^{\omega_2\rho_{1k}\pi} - \omega_3 e^{\omega_3\rho_{1k}\pi} = 0,$$

and

$$e^{\omega_3\rho_{1k}\pi} - \omega_2 e^{\omega_1\rho_{1k}\pi} - \omega_3 e^{\omega_3\rho_{1k}\pi} = 0,$$

the first being Hopkins' characteristic equation, and the second being obtained by multiplying the first by $-\omega_2$ and rearranging the terms. From these two equations we get

$$1 : -\omega_2 : -\omega_3 = e^{2\omega_3\rho_{1k}\pi} - e^{-\omega_3\rho_{1k}\pi} : e^{2\omega_3\rho_{1k}\pi} - e^{-\omega_3\rho_{1k}\pi} : e^{2\omega_1\rho_{1k}\pi} - e^{-\omega_1\rho_{1k}\pi}.$$

Consequently if in (21) we replace ρ by ρ_{1k} and remove the proper factor independent of x, we get what we shall take as one set of characteristic functions, namely $u_{1k}(x) = \delta_3(\rho_{1k}x)$. These are Hopkins' characteristic functions. If we transform by means of the transformation

$$x' = \omega_2\pi - x \tag{23}$$

to the point $\omega_2\pi$ as a new origin, we find by steps similar to those above that the rest of our characteristic functions are given by $u_{2k}(x) = \delta_3(\rho_{1k}x') = \delta_3[\rho_{1k}(\omega_2\pi - x)]$.

From Part I we know that if the series

$$\sum_1^\infty a_{1k}\, u_{1k}(x) \tag{24}$$

converges uniformly in some segment of the axis of reals from 0 to π, it represents a function analytic in x and of the form $x^2\phi(x^3)$, $\phi(x^3)$ being a convergent power series in x^3, and that the series also converges uniformly in an equilateral triangle centered at $x=0$ and having one vertex on the axis of reals between 0 and π. Noting that the transformation (23) consists of

a parallel shift $x''=x-\omega_2\pi$ followed by a rotation through 180°, $x'=-x''$, we see that if a series

$$\sum_1^\infty a_{2k}\, u_{2k}(x) \tag{25}$$

converges uniformly in some segment of the straight line between $\omega_2\pi$ and $-\omega_3\pi$, it represents a function of the form $(\omega_2\pi-x)^2\ \psi[(\omega_2\pi-x)^3]$, where $\psi[(\omega_2\pi-x)^3]$ is a convergent power series in $(\omega_2\pi-x)^3$, and the series also converges uniformly in an equilateral triangle centered at $x=\omega_2\pi$ and having one vertex on the segment from $\omega_2\pi$ to $-\omega_3\pi$. If the triangle centered at $x=0$, which we call triangle I, and the triangle centered at $x=\omega_2\pi$, which we call triangle II, are so large as to have one side in common, it is conceivable that on this common side a series of the type

$$\sum_1^\infty [a_{1k}u_{1k}(x) + a_{2k}u_{2k}(x)] \tag{26}$$

may represent functions of a wider class than those represented by either series (24) or (25) individually or than a linear combination of functions representable by series (24) and (25).

Let, therefore, triangles I and II be so large that they have a common side C, which will be the segment from π to $-\omega_3\pi$. Putting $f(x)$ equal to series (26), how shall we determine formally the a's? To answer this question we proceed to a definition of adjoints.

Write $L(u)=u'''+\rho^3u$ and $M(v)=-v'''+\rho^3v$. Then

$$\int_C [vL(u) - uM(v)]dx = \int_C [vu''' - uv''']dx = [uv'' - u'v' + u''v]_C.$$

If we require that $u(\pi)=u(-\omega_3\pi)=0$, $u''(\pi)=u''(-\omega_3\pi)$, and $v'(\pi)=v'(-\omega_3\pi)=0$, $v(\pi)=v(-\omega_3\pi)$, we have

$$\int_C [vL(u) - uM(v)]dx = 0.$$

If we require also that u satisfy $L(u)=0$, u will be a characteristic function $u_{1i}(x)$ or $u_{2i}(x)$. The functions v satisfying the above conditions at π and $-\omega_3\pi$ and also the differential equation $M(v)=0$ we define as the adjoints of the u characteristic functions. Mere formal manipulations suffice to show that the characteristic numbers for the v differential system are the same as those for the u system. Adopting an obvious notation for the v characteristic functions, we have

$$\int_C [v_{2j}L(u_{1i}) - u_{1i}M(v_{2j})]dx = 0.$$

But

$$\int_C [v_{2j}L(u_{1i}) - u_{1i}M(v_{2j})]dx = (\rho_{1i}^3 - \rho_{2j}^3)\int_C u_{1i}v_{2j}dx.$$

Hence

$$\int_C u_{1i}v_{2j}dx = 0 \qquad \text{for all } i \text{ and } j \text{ ; and similarly}$$

$$\int_C u_{2i}v_{1j}dx = 0 \qquad \text{for all } i \text{ and } j,$$

(27)

$$\int_C u_{1i}v_{1j}dx = 0 \qquad \text{if } i \neq j, \text{ and}$$

$$\int_C u_{2i}v_{2j}dx = 0 \qquad \text{if } i \neq j.$$

All integrations over C are to be made from $-\omega_2\pi$ to π. The functions v are found to be given by $v_{1k}(x) = \delta_1[\rho_{1k}(\omega_2\pi - x)]$ and $v_{2k}(x) = \delta_1(\rho_{1k}x)$, except possibly for factors independent of x, of which we take no account. The relations (27) enable us to determine formally values for the a's in series (26).

We proceed to study series (26) under the hypothesis that it converges uniformly in some segment of C. Writing $x = \xi + (\pi - \xi)i/3^{1/2}$, which yields x on the line of C if ξ is real, we have

$$\begin{aligned}
(28) \qquad u_{1k}(x) = {} & e^{-\rho_k\xi}\left[\cos\rho_k\frac{\pi-\xi}{3^{1/2}} - i\sin\rho_k\frac{\pi-\xi}{3^{1/2}}\right] \\
& - e^{\rho_k(\xi-\pi/2)}\left[\cos\left(\frac{\pi}{3} + \frac{\rho_k\xi}{3^{1/2}} + \frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)\right. \\
& \left. + i\sin\left(\frac{\pi}{3} + \frac{\rho_k\xi}{3^{1/2}} + \frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)\right] \\
& - e^{\rho_k\pi/2}\left[\cos\left(\frac{\pi}{3} + \frac{2\rho_k\xi}{3^{1/2}} + \frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)\right. \\
& \left. - i\sin\left(\frac{\pi}{3} + \frac{2\rho_k\xi}{3^{1/2}} - \frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)\right],
\end{aligned}$$

and

$$u_{2k}(x) = e^{\rho_k(\xi-\pi/2)}\left[\cos\left(\frac{\rho_k\pi}{2\cdot 3^{1/2}}+\frac{\rho_k\xi}{3^{1/2}}\right)-i\sin\left(\frac{\rho_k\pi}{2\cdot 3^{1/2}}+\frac{\rho_k\xi}{3^{1/2}}\right)\right]$$

(29)
$$-e^{-\rho_k\xi}\left[\cos\left(\frac{\pi}{3}+\frac{\rho_k\pi}{3^{1/2}}-\frac{\rho_k\xi}{3^{1/2}}\right)+i\sin\left(\frac{\pi}{3}+\frac{\rho_k\pi}{3^{1/2}}-\frac{\rho_k\xi}{3^{1/2}}\right)\right]$$
$$-e^{\rho_k\pi/2}\left[\cos\left(\frac{\pi}{3}+\frac{\rho_k\pi}{2\cdot 3^{1/2}}-\frac{2\rho_k\xi}{3^{1/2}}\right)-i\sin\left(\frac{\pi}{3}+\frac{\rho_k\pi}{2\cdot 3^{1/2}}-\frac{2\rho_k\xi}{3^{1/2}}\right)\right].$$

Here we have used ρ_k to stand for ρ_{1k}, and we suppose not only that $-\pi/2<\xi<\pi$, so that x will be on C, but also that ξ is so restricted that x is within the range of assumed uniform convergence of (26).

Take k so large that there will be a value ξ' of ξ in its restricted range such that

$$\cos\left(\frac{\pi}{3}+\frac{\rho_k\pi}{2\cdot 3^{1/2}}-\frac{2\rho_k\xi}{3^{1/2}}\right)=1.$$

If x' is the value of x which corresponds to ξ', we find

$$u_{1k}(x') = e^{\rho_k\pi/2}[\omega_2+E_{1k}] \quad\text{and}\quad u_{2k}(x') = e^{\rho_k\pi/2}[-1+E_{2k}],$$

where E_{1k} and E_{2k} both can be made as small numerically as desired by taking k sufficiently large. Consequently

(30) $$a_{1k}(\omega_2+E_{1k})+a_{2k}(-1+E_{2k}) = e^{-\rho_k\pi/2}[a_{1k}u_{1k}(x')+a_{2k}u_{2k}(x')].$$

If k is large enough, we can repeat the work of this paragraph with another value ξ'' of ξ, such that

$$\cos\left(\frac{\pi}{3}+\frac{2\rho_k\xi''}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)=1,$$

and if x'' is the corresponding value of x, we shall obtain

(30′) $$a_{1k}(-1+E^1_{1k})+a_{2k}(\omega_2+E^1_{2k}) = e^{-\rho_k\pi/2}[a_{1k}u_{1k}(x'')+a_{2k}u_{2k}(x'')].$$

Equations (30) and (30′) are to be solved as non-homogeneous linear equations for the a_{1k} and a_{2k} which appear in their left hand members. If the solution is made in the usual way by determinants, we find a_{1k} given by a quotient of two determinants times $e^{-\rho_k\pi/2}$. The determinant in the denominator of this quotient is seen to have a numerical value greater than some positive constant, while the determinant in the numerator is less numerically than some positive constant, since $|a_{1k}u_{1k}(x)+a_{2k}u_{2k}(x)|<g$, where g is independent of both x and k, holds for both $x=x'$ and $x=x''$. Similar remarks are valid for the solution for a_{2k}. Thus we have proved the

LEMMA. *If series* (26) *converges uniformly in some segment of C, then* $|a_{1k}| < h_1 e^{-\rho_k \pi/2}$ *and* $|a_{2k}| < h_2 e^{-\rho_k \pi/2}$, h_1 *and* h_2 *being independent of* k.

From this lemma we may prove the following theorem by the method of reasoning used in the work immediately following the lemma of Part I.

THEOREM 1. *If series* (26) *converges uniformly in some segment of C, then series* (24) *converges uniformly in the interior of triangle* I *and represents a function analytic in* x *and of the form* $x^2\phi(x^3)$, *and series* (25) *converges uniformly in the interior of triangle* II *and represents a function analytic in* x *and of the form* $(\omega_2\pi - x)^2\psi[(\omega_2\pi - x)^3]$.

Let us now write purely formally $f(x) = \sum_1^\infty [a_{1k}u_{1k}(x) + a_{2k}u_{2k}(x)]$, and determine the a's by means of relations (27). We shall suppose that $f(x)$ is continuous together with its first two derivatives on the closed segment C. We find

$$\int_C f(x)v_{1k}(x)dx = a_{1k}\int_C u_{1k}(x)v_{1k}(x)dx, \tag{31}$$

and

$$\int_C f(x)v_{2k}(x)dx = a_{2k}\int_C u_{2k}(x)v_{2k}(x)dx. \tag{32}$$

Carrying out the integrations appearing on the right hand sides of these equations explicitly, we find

$$\int_C u_{1k}(x)v_{1k}(x)dx = \int_C u_{2k}(x)v_{2k}(x)dx = (\omega_2 - \omega_3)\pi\delta_3(-\rho_k\pi). \tag{33}$$

Also, making two integrations by parts,

$$\int_C f(x)v_{1k}(x)dx = -\frac{1}{\rho_k}[\omega_3 f(\pi) + f(-\omega_3\pi)]\delta_2(\rho_k\pi) + \frac{1}{\rho_k^2}\int_C \delta_3[\rho_k(\omega_2\pi - x)]f''(x)dx,$$

and

$$\int_C f(x)v_{2k}(x)dx = -\frac{1}{\rho_k}[f(\pi) + \omega_3 f(-\omega_3\pi)]\delta_2(\rho_k\pi) + \frac{1}{\rho_k^2}\int_C \delta_3(\rho_k x)f''(x)dx.$$

We may, therefore, write

$$(31') \quad a_{1k} = \frac{\omega_3 f(\pi) + f(-\omega_3\pi)}{(\omega_3 - \omega_2)\pi} \cdot \frac{\delta_2(\rho_k\pi)}{\rho_k\delta_3(-\rho_k\pi)} + \frac{1}{(\omega_2 - \omega_3)\pi\rho_k^2\delta_3(-\rho_k\pi)} \int_C \delta_3[\rho_k(\omega_2\pi - x)]f''(x)dx,$$

and

$$(32') \quad a_{2k} = \frac{f(\pi) + \omega_3 f(-\omega_3\pi)}{(\omega_3 - \omega_2)\pi} \cdot \frac{\delta_2(\rho_k\pi)}{\rho_k\delta_3(-\rho_k\pi)} + \frac{1}{(\omega_2 - \omega_3)\pi\rho_k^2\delta_3(-\rho_k\pi)} \int_C \delta_3(\rho_k x)f''(x)dx.$$

We now insert these expressions for the a's into series (24) and (25), and test these for convergence on C. Before actually doing this, however, we investigate the sizes of a_{1k} and a_{2k}. Consider the integral in (31'). It contains in its integrand the sum of three exponentials, no one of which at any point of C exceeds $e^{\rho_k\pi/2}$ in numerical value. Hence

$$\left| \int_C \delta_3[\rho_k(\omega_2\pi - x)]f''(x)dx \right| < Me^{\rho_k\pi/2},$$

M being independent of k. Also

$$\frac{1}{\delta_3(-\rho_k\pi)} = e^{-\rho_k\pi}[1 + \epsilon(\rho_k)],$$

where $\epsilon(\rho_k)$ has the limit zero as $k \to \infty$. Hence the term of (31') which contains the C integral can be written $(1/\rho_k^2)e^{-\rho_k\pi/2}\, M(\rho_k)$, where $M(\rho_k)$ is bounded for k large. Now

$$(34) \quad \frac{\delta_2(\rho_k\pi)}{\delta_3(-\rho_k\pi)} = e^{-\rho_k\pi/2}\left[-2\cos\left(\frac{\rho_k\pi \cdot 3^{1/2}}{2} - \frac{\pi}{3}\right) + \frac{\omega_3 e^{-2\omega_3\rho_k\pi + \rho_k\pi/2} + \omega_2 e^{-2\omega_3\rho_k\pi + \rho_k\pi/2} - e^{-\rho_k\pi/2}}{\delta_3(-\rho_k\pi)}\right].$$

Hence (31') can be written

$$(35) \quad a_{1k} = e^{-\rho_k\pi/2}\left[2\frac{\omega_3 f(\pi) + f(-\omega_3\pi)}{(\omega_2 - \omega_3)\pi\rho_k}\cos\left(\frac{\rho_k\pi \cdot 3^{1/2}}{2} - \frac{\pi}{3}\right) + \frac{M'(\rho_k)}{\rho_k^2}\right],$$

where $M'(\rho_k)$ is bounded for k large.

But from (28) we have on C

$$(36)\quad u_{1k}(x) = e^{\rho_k\pi/2}\left[-\cos\left(\frac{\pi}{3}+\frac{2\rho_k\xi}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right) + i\sin\left(\frac{\pi}{3}+\frac{2\rho_k\xi}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)+\epsilon_k(\xi)\right],$$

where $\epsilon_k(\xi)$ tends to zero uniformly as an exponential function when $k\to\infty$ if x is restricted to a closed interval on C which does not include the end points of C. From (35) and (36) it is clear that series (24) will converge uniformly on those segments of C on which

$$(37)\quad \sum_1^\infty \frac{1}{\rho_k}\cos\left(\frac{\rho_k\pi\cdot 3^{1/2}}{2}-\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}+\frac{2\rho_k\xi}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right) \text{ and}$$
$$\sum_1^\infty \frac{1}{\rho_k}\cos\left(\frac{\rho_k\pi\cdot 3^{1/2}}{2}-\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}+\frac{2\rho_k\xi}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)$$

both converge uniformly. From the equation whose roots are ρ_k we get

$$\cos\left(\frac{\rho_k\pi\cdot 3^{1/2}}{2}-\frac{\pi}{3}\right) = (-1)^k\frac{3^{1/2}}{2}\left(1-\frac{1}{4}e^{-3\rho_k\pi}\right)^{1/2}-\frac{1}{4}e^{-3\rho_k\pi/2}.$$

It is clear, therefore, that both series (37) will converge uniformly on those segments of C on which

$$\sum_1^\infty \frac{(-1)^k}{\rho_k}\cos\left(\frac{2\rho_k\xi}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right) \quad\text{and}\quad \sum_1^\infty \frac{(-1)^k}{\rho_k}\sin\left(\frac{2\rho_k\xi}{3^{1/2}}-\frac{\rho_k\pi}{2\cdot 3^{1/2}}\right)$$

both converge uniformly. Since $\rho_k=(1/3^{1/2})(1/3+2k)+\epsilon_k$, where ϵ_k is given by

$$\sin\frac{\pi\epsilon_k\cdot 3^{1/2}}{2} = \frac{(-1)^{k+1}}{2}e^{-3\rho_k\pi/2},$$

we see by simple reductions that these series will converge uniformly on those segments of C on which

$$\sum_1^\infty \frac{(-1)^k}{\rho_k}\cos ks \quad\text{and}\quad \sum_1^\infty \frac{(-1)^k}{\rho_k}\sin ks, \qquad s=\frac{1}{3}(4\xi-\pi),$$

both converge uniformly. However, these series are known to converge uniformly except possibly in neighborhoods of the points $s=2l\pi$ and $s=(2l-1)\pi$.* These exceptional points correspond on C only to the end

* Bôcher, *Introduction to the theory of Fourier series*, Annals of Mathematics, (2), vol. 7 (1906), p. 111.

and mid-points. Similar considerations are valid for a_{2k} and $u_{2k}(x)$ on C, the work being based on (32′) and (29). We may, therefore, state

THEOREM 2. *If $f(x)$ is continuous together with its first two derivatives, and if a_{1k} and a_{2k} are given by* (31′) *and* (32′), *then series* (24), (25), *and also* (26) *converge uniformly in all closed intervals of C not containing the end or mid-points of C.*

If series (26) converges uniformly on C, and $f(x)$, the function to which it converges, has a continuous second derivative on C, then the determinations (31′) and (32′) of a_{1k} and a_{2k} will be valid. Hence series (24) and (25) will both converge uniformly in all closed intervals of C not containing the end or mid-points of C, and will represent on these intervals continuous functions of x. Referring to Theorem 1, we see that we may now state

THEOREM 3. *If series* (26) *converges uniformly on C to a function which has a continuous second derivative on C, then series* (24) *and* (25) *both converge uniformly in all closed intervals of C not containing the end or mid-points of C, series* (24) *converges uniformly in triangle* I *to an analytic function of x of the form $x^2\phi(x^3)$, and series* (25) *converges uniformly in triangle* II *to an analytic function of x of the form $(\omega_2\pi - x)^2\psi[(\omega_2\pi - x)^3]$.*

We see, therefore, that under the hypotheses of this theorem, series (26) must represent on C a function which is capable of being written in the form

$$x^2\phi(x^3) + (\omega_2\pi - x)^2\psi[(\omega_2\pi - x)^3] \tag{38}$$

on C, where $\phi(x^3)$ is an analytic function of x in triangle I and $\psi[(\omega_2\pi - x)^3]$ an analytic function of x in triangle II, both being of the forms indicated.

Let us now begin with a function $f(x)$ of the form (38) on C and such that $\phi(x^3)$ and $\psi[(\omega_2\pi - x)^3]$ both have continuous second derivatives on C. Will the series (26) with coefficients given by (31) and (32) converge to $f(x)$ on C? Let us write $x^2\phi(x^3) = \sum_1^\infty a_{1k}u_{1k}(x)$. It is known that if

$$a_{1k} = \int_0^\pi x^2\phi(x^3)\delta_3[\rho_k(\pi - x)]dx \div \int_0^\pi u_{1k}(x)\delta_3[\rho_k(\pi - x)]dx,$$

then this series will converge uniformly to $x^2\phi(x^3)$ along a certain segment of the real axis between 0 and π.* We shall prove that a_{1k} thus given equals a_{1k} as given by (31). To this end we need only prove that

* Hopkins, loc. cit.

$$\int_C f(x)v_{1k}(x)dx \int_0^\pi u_{1k}(x)\delta_3[\rho_k(\pi - x)]dx$$

$$= \int_C u_{1k}(x)v_{1k}(x)dx \int_0^\pi x^2\phi(x^3)\delta_3[\rho_k(\pi - x)]dx.$$

But

$$\int_0^\pi u_{1k}(x)\delta_3[\rho_k(\pi - x)]dx = \pi\delta_2(\rho_k\pi).$$

Hence, in view of (33), we need only prove

$$(39)\quad \delta_2(\rho_k\pi)\int_C f(x)v_{1k}(x)dx = (\omega_2 - \omega_3)\delta_3(-\rho_k\pi)\int_0^\pi x^2\phi(x^3)\delta_3[\rho_k(\pi - x)]dx.$$

But

$$\int_C f(x)v_{1k}(x)dx = \int_C \{x^2\phi(x^3) + (\omega_2\pi - x)^2\psi[(\omega_2\pi - x)^3]\}\delta_1[\rho_k(\omega_2\pi - x)]dx$$

$$= \int_C x^2\phi(x^3)\delta_1[\rho_k(\omega_2\pi - x)]dx,$$

since $(\omega_2\pi - x)^2\psi[(\omega_2\pi - x)^3]\ \delta_1[\rho_k(\omega_2\pi - x)]dx$ is the differential of a single-valued function of $(\omega_2\pi - x)^3$. Also

$$\int_C x^2\phi(x^3)\delta_1[\rho_k(\omega_2\pi - x)]dx = \left[\int_{-\omega_3\pi}^0 + \int_0^\pi\right]x^2\phi(x^3)\delta_1[\rho_k(\omega_2\pi - x)]dx$$

$$= \int_0^\pi x^2\phi(x^3)\{\delta_1[\rho_k(\omega_2\pi - x)] - \delta_1[\rho_k(\omega_2\pi + \omega_3 x)]\}dx$$

by Cauchy's integral theorem together with a change of integration variable. Hence to establish (39) we need merely prove

$$\delta_2(\rho_k\pi)\{\delta_1[\rho_k(\omega_2\pi - x)] - \delta_1[\rho_k(\omega_2\pi + \omega_3 x)]\}$$

$$= (\omega_2 - \omega_3)\delta_3(-\rho_k\pi)\delta_3[\rho_k(\pi - x)],$$

the truth of which is readily established by use of the equation defining the numbers ρ_k.

But with a_{1k} as given by (31) we know that series (24) converges uniformly along C except possibly at the end and mid-points. In view of the above it follows that (24) converges uniformly to $x^2\phi(x^3)$ on C except possibly near the end and mid-points. In a similar way we may show that with a_{2k} as given by (32), series (25) will converge uniformly to $(\omega_2\pi - x)^2\ \psi[(\omega_2\pi - x)^3]$ on C except possibly near the end and mid-points. We may, therefore, state

THEOREM 4. *If* $f(x)=x^2\phi(x^3)+(\omega_2\pi-x)^2\ \psi[(\omega_2\pi-x)^3]$ *on* C, *where* $\phi(x^3)$ *is continuous together with its first two derivatives on* C *and analytic in* x *and of the form indicated in triangle* I, *and where* $\psi[(\omega_2\pi-x)^3]$ *is continuous together with its first two derivatives on* C *and analytic in* x *and of the form indicated in triangle* II, *then the series* (26) *with the formal determinations of the coefficients will converge uniformly to* $f(x)$ *on all closed intervals of* C *not containing an end or mid-point of* C.

PART III

The type of expansion considered in Part II is one of a class in which the characteristic functions are determined by the differential equation $u'''+\rho^3u=0$ and three boundary conditions imposed at two of the points π, $-\omega_2\pi$, $-\omega_3\pi$. We shall not consider this type further, but take instead a case in which the boundary conditions bear in a symmetric fashion at these three points. We take $u(\pi)=u(-\omega_2\pi)=u(-\omega_3\pi)=0$.

Here the characteristic equation is

$$\Delta(\rho)\equiv\begin{vmatrix} e^{\omega_1\rho\pi} & e^{\omega_2\rho\pi} & e^{\omega_3\rho\pi} \\ e^{\omega_2\rho\pi} & e^{\omega_3\rho\pi} & e^{\omega_1\rho\pi} \\ e^{\omega_3\rho\pi} & e^{\omega_1\rho\pi} & e^{\omega_2\rho\pi} \end{vmatrix}=0.$$

Simple reductions give

$$\Delta(\rho)\equiv 3-e^{3\omega_1\rho\pi}-e^{3\omega_2\rho\pi}-e^{3\omega_3\rho\pi}\equiv-\delta_1(\rho\pi)\delta_2(\rho\pi)\delta_3(\rho\pi). \tag{40}$$

Accordingly, the characteristic numbers fall into three sets, all distributed along the positive axis of reals, and determined respectively by the equations

$$\delta_1(\rho\pi)=0, \qquad \delta_2(\rho\pi)=0, \qquad \delta_3(\rho\pi)=0.$$

They are given respectively by

$$\rho_{1k}=\frac{1}{3^{1/2}}(1+2k)+\epsilon_{1k}, \quad \rho_{2k}=\frac{1}{3^{1/2}}\left(\frac{5}{3}+2k\right)+\epsilon_{2k}, \quad \rho_{3k}=\frac{1}{3^{1/2}}\left(\frac{1}{3}+2k\right)+\epsilon_{3k}, \tag{41}$$

where ϵ_{ik} tends to zero as a limit when $k\to\infty$, i being 1, 2, or 3.

Use of the conditions $u(\pi)=u(-\omega_2\pi)=0$ gives

$$u(x)=e^{\omega_1\rho x}(e^{2\omega_2\rho\pi}-e^{-\omega_3\rho\pi})+e^{\omega_2\rho x}(e^{2\omega_1\rho\pi}-e^{-\omega_1\rho\pi})+e^{\omega_3\rho x}(e^{2\omega_1\rho\pi}-e^{-\omega_2\rho\pi}), \tag{42}$$

except possibly for a factor independent of x. The characteristic numbers ρ_{1k} satisfy the following pair of equations; the second is obtained from the first by a rearrangement of terms:

$$e^{\omega_1\rho\pi} + e^{\omega_2\rho\pi} + e^{\omega_3\rho\pi} = 0,$$

$$e^{\omega_2\rho\pi} + e^{\omega_3\rho\pi} + e^{\omega_1\rho\pi} = 0.$$

Hence

$$1:1:1: = e^{2\omega_3\rho_{1k}\pi} - e^{-\omega_3\rho_{1k}\pi} : e^{2\omega_1\rho_{1k}\pi} - e^{-\omega_1\rho_{1k}\pi} : e^{2\omega_2\rho_{1k}\pi} - e^{-\omega_2\rho_{1k}\pi}.$$

Thus we obtain from (42) for one set of characteristic functions $u_{1k}(x) = \delta_1(\rho_{1k}x)$, a suitable factor independent of x having been divided out of the right hand side of (42). Similarly we obtain from (42) and the equations for the other sets of characteristic numbers $u_{2k}(x) = \delta_2(\rho_{2k}x)$ and $u_{3k}(x) = \delta_3(\rho_{3k}x)$.

We have thus three sets of characteristic functions; each set consists of the complete set of characteristic functions of an expansion problem of the type considered in Part I. Let $f(x)$ be a function analytic in x at $x=0$. We may write $f(x) = \phi_1(x^3) + x\phi_2(x^3) + x^2\phi_3(x^3)$, each ϕ standing for a convergent power series in x^3. If each ϕ satisfies certain continuity conditions from 0 to π, Theorem 2 of Part I tells us that $x^{i-1}\,\phi_i(x^3)$ may be expanded in a series of type

$$\sum_{k=1}^{\infty} a_{ik}u_{ik}(x) \qquad (i = 1,2,3). \tag{43}$$

We may, therefore, state

THEOREM 1. *Subject to the continuity conditions on the interval* $(0, \pi)$ *of Theorem* 2, *Part* I, *any analytic function of* x *may be expanded in a series of the type*

$$\sum_{1}^{\infty} [a_{1k}u_{1k}(x) + a_{2k}u_{2k}(x) + a_{3k}u_{3k}(x)], \tag{44}$$

and the series will converge uniformly to the function in the interior of an equilateral triangle centered at $x=0$ *and having one vertex on the segment from* 0 *to* π.

The following theorem, which we will now prove, states essentially that a series of type (44) necessarily represents a function which is analytic in x.

THEOREM 2. *If series* (44) *converges uniformly in an interval* (α, β), $0 \leqq \alpha < \beta \leqq \pi$, *each of the three series* (43) *converges uniformly in the interior of an equilateral triangle centered at* $x=0$ *and having one vertex in the interior of the interval* (α, β).

Let x_{1k}, x_{2k}, x_{3k} be three values of x in (α, β), k being fixed, such that $u_{ik}(x_{ik})=0$, $i=1, 2, 3$. It is clear that if k is sufficiently large, three such values of x will exist. Then, calling

$$u_k(x) = a_{1k}u_{1k}(x) + a_{2k}u_{2k}(x) + a_{3k}u_{3k}(x),$$

we have

$$\begin{aligned} u_k(x_{1k}) &= a_{2k}u_{2k}(x_{1k}) + a_{3k}u_{3k}(x_{1k}), \\ u_k(x_{2k}) &= a_{1k}u_{1k}(x_{2k}) + a_{3k}u_{3k}(x_{2k}), \\ u_k(x_{3k}) &= a_{1k}u_{1k}(x_{3k}) + a_{2k}u_{2k}(x_{3k}). \end{aligned}$$

Calling

$$\Delta \equiv \begin{vmatrix} 0 & u_{2k}(x_{1k}) & u_{3k}(x_{1k}) \\ u_{1k}(x_{2k}) & 0 & u_{3k}(x_{2k}) \\ u_{1k}(x_{3k}) & u_{2k}(x_{3k}) & 0 \end{vmatrix},$$

we get

$$a_{1k} = \frac{1}{\Delta} \begin{vmatrix} u_k(x_{1k}) & u_{2k}(x_{1k}) & u_{3k}(x_{1k}) \\ u_k(x_{2k}) & 0 & u_{3k}(x_{2k}) \\ u_k(x_{3k}) & u_{2k}(x_{3k}) & 0 \end{vmatrix}, \tag{45}$$

with similar formulas for a_{2k} and a_{3k}. We shall use these expressions to determine the sizes of the a's.

We have

$$u_{1k}(x_{2k}) = e^{-\rho_{1k}x_{2k}} + 2e^{\rho_{1k}x_{2k}/2} \cos\left(\frac{3^{1/2}}{2}\rho_{1k}x_{2k}\right).$$

But from (41)

$$\cos\left(\frac{3^{1/2}}{2}\rho_{1k}x_{2k}\right) = \cos\left[\frac{3^{1/2}}{2}\rho_{2k}x_{2k} - \frac{x_{2k}}{3} + \frac{3^{1/2}}{2}x_{2k}(\epsilon_{1k} - \epsilon_{2k})\right].$$

Also, from $u_{2k}(x_{2k})=0$, we get

$$\cos\left(-\frac{\pi}{3} + \frac{3^{1/2}}{2}\rho_{2k}x_{2k}\right) = \frac{1}{2}e^{-3\rho_{2k}x_{2k}/2},$$

whence

$$\sin\left(-\frac{\pi}{3} + \frac{3^{1/2}}{2}\rho_{2k}x_{2k}\right) = \left(1 - \frac{1}{4}e^{-3\rho_{2k}x_{2k}/2}\right)^{1/2}.$$

We may choose the positive sign before the radical by selecting x_{2k} properly. Hence

$$\left(\cos\frac{3^{1/2}}{2}\rho_{1k}x_{2k}\right) = -\sin\left(\frac{\pi}{3} - \frac{x_{2k}}{3}\right) + \epsilon,$$

where ϵ is a symbol which we shall use for any quantity which can be made as small numerically as desired by taking k large enough. Hence

$$u_{1k}(x_{2k}) = e^{\rho_{1k}x_{2k}/2}\left[-2\sin\left(\frac{\pi}{3} - \frac{x_{2k}}{3}\right) + \epsilon\right].$$

In a similar way we find

$$u_{1k}(x_{3k}) = e^{\rho_{1k}x_{3k}/2}\left[2\sin\left(\frac{\pi}{3} - \frac{x_{3k}}{3}\right) + \epsilon\right],$$

$$u_{2k}(x_{1k}) = e^{\rho_{2k}x_{1k}/2}\left[-2\sin\left(\frac{\pi}{3} - \frac{x_{1k}}{3}\right) + \epsilon\right],$$

$$u_{2k}(x_{3k}) = e^{\rho_{2k}x_{3k}/2}\left[-2\sin\left(\frac{2\pi}{3} - \frac{2x_{3k}}{3}\right) + \epsilon\right],$$

$$u_{3k}(x_{1k}) = e^{\rho_{3k}x_{1k}/2}\left[2\sin\left(\frac{\pi}{3} - \frac{x_{1k}}{3}\right) + \epsilon\right],$$

$$u_{3k}(x_{2k}) = e^{\rho_{3k}x_{2k}/2}\left[2\sin\left(\frac{2\pi}{3} - \frac{2x_{2k}}{3}\right) + \epsilon\right].$$

Hence

$$\begin{aligned}\Delta = {} & e^{(\rho_{1k}x_{2k}+\rho_{2k}x_{3k}+\rho_{3k}x_{1k})/2}\left[-2\sin\left(\frac{\pi}{3} - \frac{x_{2k}}{3}\right) + \epsilon\right] \\ & \cdot\left[-2\sin\left(\frac{2\pi}{3} - \frac{2x_{3k}}{3}\right) + \epsilon\right]\left[2\sin\left(\frac{\pi}{3} - \frac{x_{1k}}{3}\right) + \epsilon\right] \\ & + e^{(\rho_{1k}x_{3k}+\rho_{2k}x_{1k}+\rho_{3k}x_{2k})/2}\left[2\sin\left(\frac{\pi}{3} - \frac{x_{3k}}{3}\right) + \epsilon\right] \\ & \cdot\left[-2\sin\left(\frac{\pi}{3} - \frac{x_{1k}}{3}\right) + \epsilon\right]\left[2\sin\left(\frac{2\pi}{3} - \frac{2x_{2k}}{3}\right) + \epsilon\right].\end{aligned}$$

By means of obvious transformations this can be written

$$\begin{aligned}\Delta = {} & 16\sin\frac{\pi - x_{1k}}{3}\sin\frac{\pi - x_{2k}}{3}\sin\frac{\pi - x_{3k}}{3}e^{(\rho_{1k}x_{1k}+\rho_{2k}x_{2k}+\rho_{3k}x_{3k})/2} \\ & \cdot\left[\begin{array}{l} e^{[\rho_{1k}(x_{2k}-x_{1k})+\rho_{2k}(x_{3k}-x_{2k})+\rho_{3k}(x_{1k}-x_{3k})]/2}\left(\cos\dfrac{\pi - x_{3k}}{3} + \epsilon\right) \\ -e^{[\rho_{1k}(x_{3k}-x_{1k})+\rho_{2k}(x_{1k}-x_{2k})+\rho_{3k}(x_{2k}-x_{3k})]/2}\left(\cos\dfrac{\pi - x_{2k}}{3} + \epsilon\right)\end{array}\right].\end{aligned}$$

Now

$$\begin{aligned}\rho_{1k}(x_{2k} - x_{1k}) + \rho_{2k}(x_{3k} - x_{2k}) + \rho_{3k}(x_{1k} - x_{3k}) \\ = \frac{2}{3\cdot 3^{1/2}}(2x_{3k} - x_{1k} - x_{2k}) + \epsilon,\end{aligned}$$

and

$$\rho_{1k}(x_{3k}-x_{1k})+\rho_{2k}(x_{1k}-x_{2k})+\rho_{3k}(x_{2k}-x_{3k}) = \frac{2}{3\cdot 3^{1/2}}(x_{1k}+x_{3k}-2x_{2k})+\epsilon.$$

We will make $2x_{3k}-x_{1k}-x_{2k}>x_{1k}+x_{3k}-2x_{2k}$ by taking $x_{2k}+x_{3k}>2x_{1k}$. To secure this we divide (α, β) into three pieces, each a closed interval and each separated from the others by some small fixed amount. We require x_{1k} to be chosen from the left hand one of these subintervals, and x_{2k} and x_{3k} from the other two. This renders the inequalities true. We will denote these three subintervals by I, II, and III in order from left to right. Evidently $\cos((\pi-x_{3k})/3)$ and $\cos((\pi-x_{2k})/3)$ are both positive. We require that x_{2k} and x_{3k} be chosen from intervals II and III respectively. This assures us that $\cos((\pi-x_{3k})/3)>\cos((\pi-x_{2k})/3)$. It is now clear, if k is large enough, that

$$|\Delta| > he^{(\rho_{1k}x_{1k}+\rho_{2k}x_{2k}+\rho_{3k}x_{3k})/2}, \tag{46}$$

where h is independent of k.

Returning to (45) we have for the determinant there written out explicitly, which we shall call N,

$$N = u_k(x_{2k})u_{2k}(x_{3k})u_{3k}(x_{1k}) + u_k(x_{3k})u_{3k}(x_{2k})u_{2k}(x_{1k}) - u_k(x_{1k})u_{2k}(x_{3k})u_{3k}(x_{2k}).$$

On account of the assumed uniform convergence of series (44), $|u_k(x_{ik})|$ is less than some number which is independent of k. Further,

$$|u_{2k}(x_{2k})| < ge^{\rho_{2k}x_{2k}/2}, \qquad |u_{3k}(x_{1k})| < ge^{\rho_{2k}x_{1k}/2},$$
$$|u_{3k}(x_{2k})| < ge^{\rho_{3k}x_{2k}/2}, \qquad |u_{2k}(x_{1k})| < ge^{\rho_{2k}x_{1k}/2},$$

the same g, which is independent of k, serving in all cases. Hence

$$|u_{2k}(x_{3k})u_{3k}(x_{1k})| < g^2e^{(\rho_{2k}x_{3k}+\rho_{3k}x_{1k})/2} < \mu e^{(\rho_{2k}x_{2k}+\rho_{3k}x_{3k})/2},$$
$$|u_{3k}(x_{2k})u_{2k}(x_{1k})| < g^2e^{(\rho_{3k}x_{2k}+\rho_{2k}x_{1k})/2} < \mu e^{(\rho_{2k}x_{2k}+\rho_{3k}x_{3k})/2},$$
$$|u_{2k}(x_{3k})u_{3k}(x_{2k})| < g^2e^{(\rho_{2k}x_{3k}+\rho_{3k}x_{2k})/2} < \mu e^{(\rho_{2k}x_{2k}+\rho_{3k}x_{3k})/2},$$

μ being independent of k. These inequalities in conjunction with (46) yield $|a_{1k}|<\nu_1 e^{-\rho_{1k}x_{1k}/2}$, where ν_1 is independent of k.

In a similar way we find $|a_{2k}|<\nu_2 e^{-\rho_{2k}x_{1k}/2}$ and $|a_{3k}|<\nu_3 e^{-\rho_{3k}x_{1k}/2}$. Moreover it is clear that by taking k sufficiently large we may choose x_{1k} so that $\beta-x_{1k}$ is less than a preassigned positive constant. The theorem now follows by an argument like that used following the lemma of Part I. Evidently the triangle of this theorem may have one vertex as near to β as is desired.

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CONCERNING POINT SETS WHICH CAN BE MADE CONNECTED BY THE ADDITION OF A SIMPLE CONTINUOUS ARC*

BY

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In their paper *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, R. L. Moore and J. R. Kline† prove that it is possible to pass a simple continuous arc through every closed and bounded set M having the property that every closed and connected subset of M is either a single point or an arc t such that no point of t, with the exception of its end points, is a limit point of $M-t$. It is clear, however, that in order that a simple continuous arc may be drawn in such a way as to contain at least one point of every maximal connected subset of a point set M, it is not necessary that the set M be of the particular type satisfying their theorem. In this paper I shall make a study of certain conditions which a point set must satisfy in order that a simple continuous arc or an open curve may be drawn in such a way that the set in question plus that arc or curve will be connected.

LEMMA I. *If M is any closed and bounded point set, then there exists a countable number of arcs $t_1, t_2, t_3, \cdots$, such that for every positive integer n, t_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$.*

Let n denote any definite positive integer. Since M is bounded, there exists a square S which encloses M; S plus its interior can be divided by a finite number of straight lines parallel and perpendicular to the bases of S into a finite number of squares plus their interiors in such a way that the diameter of each of these squares is less than $1/n$ and such that the interiors of no two of them have a point in common. Let G denote this finite set of squares (not including their interiors), and let T denote the point set obtained by adding together all the point sets of the set G. Then since the interior of every square of the set G is of diameter less than $1/n$, every maximal connected subset of M which is of diameter greater than $1/n$ must contain at least one point in common with T. Let F denote the set of all points common to M and T. From each maximal connected subset Y of F select exactly one point

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† Annals of Mathematics, (2), vol. 20 (1919), pp. 218–223.

X; and let P denote the set of all such points (X) thus selected. Then since F is closed and has no continuum of condensation, it follows that $\overline{P}$ is a closed and totally disconnected point set.* It follows, then, from the above mentioned theorem of Moore and Kline† that $\overline{P}$ is a subset of a simple continuous arc t_n. Clearly t_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$.

THEOREM 1. *If M is a closed and bounded point set, a necessary and sufficient condition that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M is that for every continuum K of M which consists of more than one point there should exist a positive number ϵ_k such that K is not the limiting set of any collection of maximal connected subsets of M each of diameter less than ϵ_k.*

(I). The condition is sufficient. For since M satisfies the conditions of Lemma I, there exists a countable set of arcs $t_1, t_2, t_3, \cdots$, having the same property with respect to M that the corresponding set of arcs in Lemma I has with respect to the set M of Lemma I. Let K_1 denote the set of points common to t_1 and M; let K_2 denote the set common to t_2 and to those maximal connected subsets of M which have no point in t_1; K_3 the set common to t_3 and to those maximal connected subsets of M which have no point in t_1+t_2; in general, let K_n denote the set of points common to t_n and to those maximal connected subsets of M which have no point in $t_1+t_2+t_3+t_4+\cdots+t_{n-1}$. Let K denote the point set $K_1+K_2+K_3+K_4+\cdots$. I will proceed to show that $\overline{K}$ contains no continuum of condensation. Suppose, on the contrary, that $\overline{K}$ contains a continuum of condensation H. Then H is also a continuum of condensation of M; and by hypothesis there exists a positive number ϵ_H such that H is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_H. Now the elements of K have been so selected that for any given positive number, say ϵ_H, there exists a positive number $\delta(\epsilon_H)$ such that for every integer $n>\delta(\epsilon_H)$, K_n contains points of only those maximal connected subsets of M which are of diameter less than ϵ_H. Let i denote an integer greater than $\delta(\epsilon_H)$. Then $\sum_{n=i+1}^{n=\infty} K_n$ contains points of only those maximal connected subsets of M which are of diameter less than ϵ_H. It follows that not every point of H is a limit point of $\sum_{n=i+1}^{n=\infty} K_n$. Let G denote the collection of point sets $K_1, K_2, K_3, \cdots, K_i$. Let

$$A = \sum_{n=1}^{n=i} K_n, \text{ and let } B = \sum_{n=i+1}^{n=\infty} K_n.$$

* In this paper wherever a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all those points which are limit points of X.

† R. L. Moore and J. R. Kline, loc. cit.

Then $K=A+B$. Let P denote a point of H which is not a limit point of B; and let C be a circle enclosing P and not enclosing or containing any point whatever of $\overline{B}$. From a theorem due to Janiszewski,* it follows that C plus its interior contains a subcontinuum D of H. Then D is a subset of the closed set $\overline{A}$. Let K_a and K_b denote any two elements of G, K_a denoting the one of lower subscript. I will show that $\overline{K}_a$ and $\overline{K}_b$ have at most a closed and totally disconnected set in common. Suppose, on the contrary, that $\overline{K}_a$ and $\overline{K}_b$ have in common a continuum t which consists of more than one point. Then t is a subset both of t_a and of t_b; hence t is an arc. Let E and F denote the end points of t. Since t is a subset of t_a, and since t_a precedes t_b, then no point of t can belong to K_b. And since no point of t except the points E and F can be a limit point of t_b-t, then no points of t except E and F can belong to $\overline{K}_b$. But by supposition, t is a subset of $\overline{K}_b$. It follows that $\overline{K}_a$ and $\overline{K}_b$ have at most a closed and totally disconnected set in common. Let U denote the set of all points (X) of $\overline{A}$ such that for some two elements K_a and K_b of G, X is common to $\overline{K}_a$ and $\overline{K}_b$. Since U is the sum of a finite number of closed and totally disconnected point sets, U itself must be closed and totally disconnected. Hence D, a continuum consisting of more than one point, cannot be a subset of U. Therefore, there exists a point P of D such that for some element K_p of G, P belongs to $\overline{K}_p$ and is not a limit point of $\overline{A}-\overline{K}_p$. It follows from the above mentioned theorem of Janiszewski's† that $\overline{K}_p$ contains a continuum l of D such that no point of l is a limit point of $\overline{A}-\overline{K}_p$. But l is a subset of t_p. Hence l is an arc, and no points of l except its end points can be limit points of $\overline{K}_p-l$. Hence if O is an interior point of l, O is not a limit point of $\overline{K}-l$. But l, by supposition, belongs to H, a continuum of condensation of $\overline{K}$. Thus the supposition that $\overline{K}$ contains a continuum of condensation leads to a contradiction.

Now from each maximal connected subset Y of $\overline{K}$ let us select exactly one point X. Let N denote the set of all the points (X) thus selected. Since $\overline{K}$ contains no continuum of condensation, it readily follows that $\overline{N}$ is a closed and totally disconnected set. It is clear that $\overline{N}$ contains at least one point of every maximal connected subset of M which is of diameter greater than 0. Let Q denote the set of all those maximal connected subsets of M which have no point in common with $\overline{N}$. Then since every maximal connected subset of Q is a single point, it follows from our hypothesis that $\overline{Q}$ is a closed and totally disconnected point set. Let R denote the point set $\overline{N}+\overline{Q}$.

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* *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912), p. 109.

† Loc. cit.

Clearly R is closed and totally disconnected; accordingly, there exists a simple continuous arc T_0 which contains R; T_0 contains at least one point of every maximal connected subset of M.

(II). The condition is also necessary. Suppose, on the contrary, that there exists a closed and bounded point set M and a simple continuous arc T such that T contains at least one point of every maximal connected subset of M, but such that M does not satisfy the condition of Theorem 1. Then M contains some continuum K consisting of more than one point and such that for every positive number ϵ, K is the limiting set of a set of maximal connected subsets of M each of diameter less than ϵ. I will show that every point of K must be a limit point of $T-K \cdot T$. For suppose K contains a point P which is not a limit point of $T-K \cdot T$. Let C be a circle having P as center and not enclosing any point of $T-K \cdot T$ and of radius less than $\frac{1}{3}$ of the diameter of K. Let r denote the radius of C. By hypothesis there exists a set L of maximal connected subsets of M each of which is of diameter less than $\frac{1}{4}r$ such that K is the limiting set of L. Since P belongs to K, there exists an element g of L which contains a point whose distance from P is less than $\frac{1}{4}r$; and since g is of diameter less than $\frac{1}{4}r$, g must lie wholly within C. But g must contain at least one point Q of T. Now since K is of diameter $\geqq 3r$, K cannot be an element of L. Hence Q does not belong to K, and therefore must belong to $T-K \cdot T$. But Q lies within C, and C, by supposition, encloses no point of $T-K \cdot T$. It follows, then, that every point of K is a limit point of $T-K \cdot T$. It is easily seen that K must be a subset of T; and since K is closed and connected and consists of more than one point, K must be an arc. And if O denotes an interior point of K, then O is not a limit point of $T-K$. But we have just shown that every point of K is a limit point of $T-K$. Thus the hypothesis that the condition of Theorem 1 is not necessary leads to a contradiction, and the theorem is proved.

Definition. A point set M will be said to satisfy Condition L provided it is true that if K is any continuum whatever consisting of more than a single point, then there exists a positive number ϵ_K such that K is not a subset of the limiting set of any collection of maximal connected subsets of M each of diameter less than ϵ_K.

THEOREM 2. *If M is any closed point set, then in order that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M it is necessary and sufficient* (1) *that there should exist a bounded portion of the plane which contains at least one point of every maximal connected subset of M, and* (2) *that M should satisfy Condition* L.

It follows by an argument similar to part (II) of the proof of Theorem 1 that the conditions are necessary. I will proceed to show that they are sufficient. By hypothesis it follows that there exists a circle C such that C plus its interior contains at least one point of every maximal connected subset of M. Let R denote the interior of C, and let N denote the set of points common to M and to $R+C$. It readily follows that N satisfies Condition L; and since N is closed and bounded, it follows from Theorem 1 that there exists an arc T which contains at least one point of every maximal connected subset of N. But every maximal connected subset of N belongs to a single maximal connected subset of M, and each maximal connected subset of M contains at least one maximal connected subset of N. If follows, then, that T contains at least one point of every maximal connected subset of M.

THEOREM 3. *If M is a closed point set which satisfies conditions* (1) *and* (2) *of Theorem* 2, *and if K is a closed and bounded subset of M having the property that every subcontinuum of K is either a single point or an arc t such that no point of t, with the exception of its end points, is a limit point of $M-t$, then there exists an arc T which contains K and which contains at least one point of every maximal connected subset of M.*

By an argument almost identical with part (I) of the proof of Theorem 1, it follows that there exists a closed, bounded, and totally disconnected point set R which contains at least one point of every maximal connected subset of M. Let N denote the point set $K+R$. Then clearly N satisfies all the conditions of the above mentioned theorem of Moore and Kline.* Accordingly, there exists a simple continuous arc T which contains N; T, then, contains K and also contains at least one point of every maximal connected subset of M.

It is interesting to note that Theorem 3 is a generalization of Moore and Kline's theorem. It reduces to their theorem in case $K=M$.

THEOREM 4. *If M is a bounded point set such that the totality of all those limit points of M which do not belong to M is a closed set, then in order that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M it is necessary and sufficient that M should satisfy Condition* L.

That the condition is necessary follows by an argument identical with part (II) of the proof of Theorem 1. I shall show that the condition is sufficient. Let M' denote the totality of all those limit points of M which M

* Loc. cit.

does not contain. For any definite positive integer n, let the sets S, G, T, F, and P be selected exactly as was done in the proof of Lemma I. Then $\overline{P}$ is totally disconnected. For suppose $\overline{P}$ contains a continuum H consisting of more than a single point. Then every point of H is a limit point of a set of points of P which belong to H. And since M' is closed, it readily follows from Janiszewski's theorem mentioned above that H contains a continuum D which consists of more than one point and which is a subset of M. Since D is a subset of a finite number of arcs, then D must contain at least one arc t such that only the end points of t are limit points of $T-t$. But since t belongs to only one maximal connected subset of F, then P contains only one point at most of t. And since only the end points of t can be limit points of P, $\overline{P}$ can contain at most three points of t. Thus the supposition that $\overline{P}$ is not totally disconnected leads to a contradiction. It follows, then, that there exists a simple continuous arc t_n which contains $\overline{P}$, and therefore contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Hence, there exists a countable set of arcs $t_1, t_2, t_3, \cdots$, such that for every positive integer n, t_n contains at least one point of every maximal connected subset of M of diameter greater than $1/n$.

Now let the sets $K_1, K_2, K_3, \cdots, K, N$, and R be selected exactly as in the proof of Theorem 1. It can then be shown that R is totally disconnected. For suppose R contains a continuum H consisting of more than one point. Then either (1) H belongs wholly to M', or (2) H contains a subcontinuum D which belongs wholly to M and which consists of more than a single point. In either case, H is a continuum of condensation of $\overline{K}$, and either of the two cases can be shown to lead to a contradiction by the same method as was used in the proof of Theorem 1 to show that the set $\overline{K}$ contained no continuum of condensation. It follows, then, that R is closed and totally disconnected; consequently, there exists a simple continuous arc which contains R and which therefore contains at least one point of every maximal connected subset of M.

THEOREM 5. *In order that a closed point set M (which is not itself an open curve) should be a subset of an open curve, it is necessary and sufficient* (1) *that every subcontinuum of M should be either a single point or a set t such that t is either an arc or a ray of an open curve having the property that no point of t, with the exception of its end point (s), is a limit point of $M-t$, and* (2) *that if M contains two rays r_1 and r_2, then $M-(r_1+r_2)$ is a bounded point set.*

The conditions are evidently necessary. I shall show that they are sufficient. There exists a circle C with center O such that C plus its interior contains no point of M. By an inversion of the whole plane about the circle

C, M is thrown into a bounded point set M^* which is closed except possibly for the point O. It is easily shown that the image under this inversion of every arc t of M is an arc t^* of M^*, and that the image of every ray r of M is an arc minus one end point in M^*, that end point in every case being the point O itself. Since M contains not more than two mutually exclusive rays, then O is an end point of not more than two arcs of M^*+O which have in common only the point O; and if O is an end point of two such arcs, i.e., if O is an interior point of any arc of M^*+O, then O is not a limit point of M^*+O minus that maximal connected subset of M^*+O to which O belongs. It readily follows, then, that M^*+O is a closed and bounded point set which satisfies all the conditions of Moore and Kline's theorem quoted above. Accordingly, there exists a simple continuous arc t which contains M^*+O. Let A and B denote the extremities of t. There exists an arc t_0 from A to B having only the points A and B in common with t. Let l^* denote the simple closed curve $t+t_0$. It can easily be shown that the point set of which l^*-O is the image is an open curve which contains M.

THEOREM 6. *If M is a closed point set (bounded or not), then in order that there should exist an open curve which contains at least one point of every maximal connected subset of M it is necessary and sufficient that M should satisfy Condition* L.

That the condition is necessary follows by an argument almost identical with part (II) of the proof of Theorem 1. I shall proceed to show that it is sufficient.

Proof I (depending on Theorem 4). Let C denote a circle having center O and not enclosing or containing any point of M. By an inversion of the plane about the circle C, M is thrown into a bounded point set M^* which is closed except possibly for the point O. I shall show that M^* satisfies Condition L. Suppose the contrary is true; then there exists a continuum K consisting of more than one point and such that for every positive number ϵ, K is the limiting set of a set of maximal connected subsets of M^* each of diameter less than ϵ. Let P be a point of K which is different from the point O, and let J^* be a circle having P as center and not containing or enclosing O. It is a consequence of Janiszewski's theorem† that J^* plus its interior contains a subcontinuum H^* of K which consists of more than one point; H^* does not contain O. Hence H, the point set of which H^* is the image under this inversion, is a bounded point set. Let I^* denote the interior of J^*, and let J and I denote the point sets of which J^* and I^* respectively are

† See Janiszewski, loc. cit.

the images. Let G_1^* denote a set of maximal connected subsets of M^* each of which has a point within J^* and is of diameter less than 1 and such that H^* is a part of the limiting set of G_1^*; let G_2^* denote a corresponding set having H^* as a part of its limiting set and such that each element of G_2^* has a point in I^* and is of diameter less than $\frac{1}{2}$; let G_3^*, G_4^*, $\cdots$ denote corresponding sets for the numbers $\frac{1}{3}$, $\frac{1}{4}$, $\cdots$. Let G_1, G_2, G_3, $\cdots$ denote the point sets of which G_1^*, G_2^*, G_3^*, $\cdots$ are the images. Then H is a part of the limiting set of each of the sets G_1, G_2, G_3, $\cdots$. But since M satisfies Condition L, there exists a positive number ϵ_H such that H is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_H. Then for every positive integer n, G_n must contain at least one element g_n which is of diameter $\geqq \epsilon_H$. From each set G_i, select one such element g_i, and let B denote the sequence of sets $g_1, g_2, g_3, \cdots$ thus obtained. Since every element of B contains at least one point in the bounded point set I, it follows that the sequence B contains some subsequence A which has a sequential limiting set $\bar{g}$ which is of diameter $\geqq \epsilon_H$. But A^*, the image of A, has the property that for every positive number ϵ, there are not more than a finite number of elements of A^* of diameter greater than ϵ. Hence, the limiting set $\bar{g}^*$ of A^* must consist of only a single point; but $\bar{g}^*$ is the image of $\bar{g}$, a point set of diameter $\geqq \epsilon_H$. Thus the supposition that M^* does not satisfy Condition L leads to a contradiction. Then since M^* satisfies Condition L and lacks only the point O of being closed, it follows by Theorem 4 that there exists a simple continuous arc t which contains at least one point of every maximal connected subset of M^*. Let X and Y denote the extremities of t. There exists an arc t_0 from X to Y which has only the points X and Y in common with t. Let l^* denote the simple closed curve $t+t_0$. Now if l^* contains the point O, it can readily be shown that the point set of which l^*-O is the image is an open curve which contains at least one point of every maximal connected subset of M. In case l^* does not contain O, then M must satisfy all the conditions of Theorem 2, and with the aid of that theorem, Theorem 6 can easily be established in this case.

Proof II (depending on Theorem 5). I will indicate how the condition may be proved sufficient using methods very similar to those used in the proof of Theorem 1. For any definite positive integer n, let the whole plane be divided by a countable infinity of horizontal and vertical straight lines into a countable number of squares plus their interiors in such a way that the interiors of no two of these squares have a point in common and so that the diameter of each of them is less than $1/n$. Let G denote this countable set of squares (not including their interiors) and let T denote the point set obtained by adding together all the point sets of the set G. Let K denote

the point set common to T and M. From each maximal connected subset Y of K, select exactly one point X. Let P denote the set of all such points (X) thus selected. It can readily be shown, then, that $\overline{P}$ is a closed and totally disconnected point set. Then from Theorem 5 it follows that there exists an open curve l_n which contains $\overline{P}$; l_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Hence, there exists a countable number of open curves $l_1, l_2, l_3, \cdots$, such that for every positive integer n, l_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Then by an argument almost identical with that used in the proof of Theorem 1, using this countable set of open curves instead of a countable set of arcs as in that case, it follows that M contains a closed and totally disconnected point set R which contains at least one point of every maximal connected subset of M. Then, by Theorem 5, there exists an open curve l which contains R; l satisfies all the conditions of the open curve required in the statement of Theorem 6.

THEOREM 7. *If M is any continuum whatever, and K is a closed and totally disconnected subset of M, then the point set $M-K$ satisfies Condition* L.

Suppose $M-K$ does not satisfy Condition L. Then there exists a continuum H consisting of more than one point such that for every positive number ϵ, H is a part of the limiting set of a set of maximal connected subsets of $M-K$ each of diameter less than ϵ. Since K is totally disconnected, there exists a point P of H which does not belong to K. Let C be a circle having P as center and not enclosing or containing any point of K. Let r denote the radius of C. Let N denote a set of maximal connected subsets of $M-K$ each of diameter less than $\frac{1}{4}r$, which has H as a part of its limiting set. Then some element g of N contains a point Q whose distance from P is less than $\frac{1}{4}r$; and since g is of diameter less than $\frac{1}{4}r$, $\bar{g}$ must lie wholly within C. But $\bar{g}$ has a point in K,* and C encloses no point of K. Thus the supposition that $M-K$ does not satisfy Condition L leads to a contradiction.

THEOREM 8. *If M is any continuum whatever, and K is a bounded [unbounded], closed, and totally disconnected subset of M, then there exists an arc [open curve] which contains at least one point of every maximal connected subset of the point set $M-K$.*

* By virtue of the following theorem: *If K is any closed subset of a continuum M and g is any bounded maximal connected subset of $M-K$, then K contains at least one point of g.*

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THE NOTION OF THE GREEN'S FUNCTION IN THE THEORY OF INTEGRO-DIFFERENTIAL EQUATIONS*

BY

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INTRODUCTION

In 1835 Duhamel in his thermo-mechanical studies discussed the integro-differential system†

$$\frac{d^2u}{dx^2} + \rho^2 u = c\rho^2 x \int_a^b u(x)x\,dx,$$

$$u(a) = 0, \quad u(b) = 0,$$

where ρ denotes a parameter and c a given constant. Since then the problems of this kind do not seem to have attracted much attention, in spite of, or perhaps because of, the great development of the general theory of integral and integro-differential equations by Volterra, Fredholm, Hilbert and their followers. In 1914, however, there appeared an interesting paper by L. Lichtenstein‡, which contains an independent treatment of the boundary problem

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + [q(x) + \lambda k(x)]y(x) + \lambda \int_0^\pi M(x,\xi)y(\xi)d\xi = h(x),$$

$$y(0) = 0, \quad y(\pi) = 0,$$

by means of the theory of quadratic forms in infinitely many variables. It was remarked by the author in 1917§ (without knowledge of Lichtenstein's

* Presented to the Society, January 1, 1926; received by the editors in October, 1926. Part IV added in May, 1927.

† *Sur les phénomènes thermo-mécaniques*, Journal de l'Ecole Polytechnique, vol. 15 (1835), cahier 25, pp. 1–57; pp. 44–48.

‡ *Ueber eine Integro-Differential Gleichung und die Entwickelung willkürlicher Funktionen nach deren Eigenfunktionen*, Schwarz's Festschrift, Berlin, 1914, pp. 274–285. Cf. also L. Koschmieder, *Anwendung der Integralgleichungen auf eine thermo-elastische Aufgabe*, Crelle's Journal, vol. 143 (1913), pp. 285–2?3.

§ *On some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function*, Petrograd, 1917 (in Russian). This paper contains as special cases most of the results of M. Stone's important paper *A comparison of the series of Fourier and Birkhoff*, these Transactions, vol. 28 (1926), pp. 695–761, concerning the theory of equiconvergence and convergence of Birkhoff series and obtained by Stone without knowledge of author's paper. The theory of the derived Birkhoff series successfully treated by Stone was not touched in our paper in question.

paper) that Duhamel's problem may yield to treatment by the method used by G. D. Birkhoff* and by the author (loc. cit.) in the theory of ordinary differential boundary problems. Some special cases of integro-differential boundary problems (mostly with constant coefficients) have been recently treated.†

The present paper sketches a more or less general theory of the integro-differential boundary problem originated by Duhamel. The characteristic feature of the paper lies in the constant use of the notion (which appears to be new) of the Green's function of the integro-differential problem. This function is closely related to the Green's function of a certain corresponding differential problem. It is found that the treatment of the integro-differential problem requires no methods other than those which have been used in the treatment of the differential boundary problem; from certain points of view the two problems are equivalent. The general theory as developed in Parts I–III admits of an interesting application to Fredholm integral equations with discontinuous kernels (Part IV).

A detailed discussion of the differential boundary problem and of the properties of the corresponding Green's function is embodied in the previous papers by the author.‡ The second of these (reference to which will be indicated merely by the symbol "D") contains the proofs of some propositions which are merely stated in the present paper.

I. Definition of the Green's function

1. We consider the integro-differential equation

$$(\bigstar) \qquad u^{(k)}(x) + r_1(x)u^{(k-1)}(x) + \cdots + r_k(x)u(x) = r(x)\sum_{\sigma=0}^{m}\int_a^b u^{(\sigma)}(\xi)\,d_\xi R_\sigma(x,\xi)$$

in which the integrals are of the Stieltjes type, and in which the functions $R_\sigma(x, \xi)$, together with their partial derivatives (to a certain order) with respect to ξ are continuous in ξ, except along a finite number of lines

$$\xi = \text{constant} = \alpha$$

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* *Boundary value and expansion problems of ordinary linear differential equations*, these Transactions, vol. 9 (1908), pp. 373–395.

† W. Jaroshek, *Entwickelung willkürlicher Funktionen* (Diss.), Breslau, 1918; H. Laudien, *Entwickelung willkürlicher Funktionen bei einem thermo-elastischen Problem*, Crelle's Journal, vol. 148 (1918), pp. 79–87; A. Kneser, *Die Integralgleichungen*, 2d edition, Braunschweig, 1922, pp. 199–214; R. Krzenziessa, *Thermo-elastischen Randwertaufgaben*, Mathematische Zeitschrift, vol. 25 (1926), pp. 209–260; the last paper discusses some cases of variable coefficients, which are reducible to Bessel functions.

‡ Loc. cit. and *Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions*, Mathematische Zeitschrift.

and possibly, along the line

$$\xi = x.$$

The discontinuities are all to be of the first kind. Under these hypotheses the equation (★) may be reduced by a suitable number of integrations by parts to the form

$$\text{(I)} \qquad L(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x) + \int_a^b h(x,\xi)u(\xi)d\xi$$

where $L(u)$ is a linear differential operator

$$\text{(1)} \qquad L(u) \equiv u^{(n)} + p_1(x)u^{(n-1)} + \cdots + p_n(x)u,$$

the $l_j(u)$ are linear forms in u and its derivatives taken at the points a, b, α, and $\phi_j(x)$, $h(x, \xi)$ are known functions.

In the following we confine the discussion to the equation (I) either in the form above or in the form

$$\text{(I')} \qquad L^{(0)}(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x) + \int_a^b h(x,\xi)u(\xi)d\xi + L^{(1)}(u),$$

where $L^{(1)}(u)$ is any linear differential operator of order $(n-1)$,

$$\text{(2)} \qquad L^{(1)}(u) \equiv p_1^{(1)}(x)u^{(n-1)} + \cdots + p_n^{(1)}(x)u,$$

and

$$L^{(0)}(u) \equiv L(u) - L^{(1)}(u).$$

The boundary conditions to be imposed on the unknown function $u(x)$ are

$$\text{(II)} \qquad L_i(u) = 0 \qquad (i = 1, 2, \cdots, n)$$

where

$$\text{(3)} \qquad L_i(u) \equiv A_i(u) + B_i(u) + \int_a^b \alpha_i(x)u(x)dx,$$

$$A_i(u) \equiv \sum_{k=1}^{n} a_{ik}u^{(k-1)}(a)\ ;\ B_i(u) \equiv \sum_{k=1}^{n} b_{ik}u^{(k-1)}(b).^*$$

The integro-differential problem consisting of (I) and (II) we shall designate as "problem (★)". The problem obtained by setting in (I')

$$\phi_j(x) = h(x,\xi) = L^{(1)}(u) \equiv 0$$

we shall designate as the "differential problem (★)."

* The boundary operators used in D, 5, are somewhat more general than those used here.

The hypotheses on the functions involved in (I′) and (II) are as follows:

i. The functions

$$p_i(x), \quad p_i^{(1)}(x)$$

possess continuous derivatives to order $(n-i)$ on (a, b).

ii. The functions

$$f(x), \quad \phi_j(x) \qquad (j = 1, 2, \cdots, \mu)$$

are integrable (in the sense of Lebesgue) on (a, b).

iii. The function $h(x, \xi)$ is bounded and is integrable as a function of both variables as well as in x, for every value of ξ, and in ξ, for every value of x.

iv. The functions

$$\alpha_i(x)$$

are integrable on (a, b).

v. The operators

$$A_i(u) + B_i(u), \quad l_j(u)$$

are linearly independent, and the operators $l_j(u)$ contain no derivatives of order higher than $(n-1)$.*

2. THEOREM 1. *Under the hypotheses* (i-v), *two alternative cases are possible as follows*:

(1) *The non-homogeneous problem* (★) *admits of a uniquely determined solution*† *for an arbitrary* $f(x)$, *and this solution can be represented in form of a definite integral*

$$u(x) = \int_a^b \Gamma(x,t)f(t)dt. \tag{III}$$

The function $\Gamma(x, t)$ *is called the Green's function of the problem* (★). *It is uniquely determined at its points of continuity.*

(2) *The non-homogeneous problem* (★) *is not possible for an arbitrary* $f(x)$. *In this case the Green's function* $\Gamma(x, t)$ *does not exist, but the homogeneous problem* (★) *admits of at least one solution not identically zero on* (a, b).‡

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* This excludes the cases in which some of the operators $L_i(u)$ reduce to the integral terms only. It is obvious also that condition (v) implies the linear independence of the operators $L_i(u)$ and $l_i(u)$.

† The term "solution" is to designate a function $u(x)$ which possesses an absolutely continuous derivative of order $(n-1)$ and which satisfies (I′) (almost everywhere) and (II).

‡ In this case we shall say simply that the homogeneous problem (★) is possible. If the homogeneous problem (★) admits of no solution other than $u(x) \equiv 0$, we shall say the problem is impossible.

3. We may assume without loss of generality that the Green's function $G(x, t)$ of the differential problem (★) exists, namely that

$$\Delta(y) = \begin{vmatrix} L_1(y_1) \cdots L_1(y_n) \\ \cdots \cdots \cdots \\ L_n(y_1) \cdots L_n(y_n) \end{vmatrix} \neq 0$$

where $y_1(x), \cdots, y_n(x)$ is a fundamental system of solutions of the equation

$$L^{(0)}(y) = 0.$$

This assumption is permissible since we may alter $L^{(0)}(y)$, if necessary, by drawing into it any portion of the operator $L^{(1)}(y)$ without thereby changing the content of equation (I′).

4. It is well known (D, 5–8) that the solution of the non-homogeneous differential problem

$$L^{(0)}(u) = \phi(x), \quad L_i(u) = 0 \qquad (i = 1, 2, \cdots, n) \tag{4}$$

is given by

$$u(x) = \int_a^b G(x,t)\phi(t)dt. \tag{5}$$

Replacing $\phi(x)$ in this by the right-hand member of (I′) we obtain for that equation the form

$$u(x) = F(x) + \sum_{j=1}^{\mu} l_j(u)\Phi_j(x) + \int_a^b H(x,\xi)u(\xi)d\xi + \int_a^b G(x,t)L^{(1)}(u)_t dt^* \tag{6}$$

where

$$F(x) = \int_a^b G(x,t)f(t)dt\ ;\ \Phi_j(x) = \int_a^b G(x,t)\phi_j(t)dt \tag{7}$$

and

$$H(x,\xi) = \int_a^b G(x,t)h(t,\xi)dt. \tag{8}$$

Under the hypotheses concerning the functions $p_i(x)$, the Green's function $G(x, t)$, as function of t possesses a continuous derivative of order $(n-2)$ and a derivative of order $(n-1)$ which is continuous except along the line $x=t$, where it has a discontinuity of the first kind (D, 11). By a suitable number of integrations by parts the final integral in (6) may be thrown into the form

$$\int_a^b H'(x,t)u(t)dt + l(u,x)$$

* The subscript t on $L^{(1)}$ is used here to indicate the variable in which the operator is written.

where $l(u, x)$ is a linear form in $u(x)$ and its derivatives to order $(n-2)$ taken at the points $x=a$ and $x=b$, the coefficients of the form being functions of x. Portions of this form may possibly be eliminated by the use of relations (II), and others may be expressible in terms of the quantities

$$l_j(u) \qquad (j = 1,2, \cdots ,\mu).$$

Thus we obtain finally

$$u(x) = F(x) + \sum_{j=1}^{\nu} l_j(u)\Phi_j'(x) + \int_a^b K'(x,\xi)u(\xi)d\xi \qquad (\nu \geqq \mu), \tag{9}$$

where the operators

$$l_j(u) \qquad (j = \mu + 1, \cdots ,\nu)$$

are of the same type as those in (I′) and the set

$$l_1(u), \quad l_2(u), \cdots , l_\nu(u) \tag{10}$$

may be considered as linearly independent and independent of $L_i(u)$. In other words, if $u(x)$ is considered arbitrary subject to (II), the quantities (10) may assume any prescribed set of values.

The functions $\Phi_j'(x)$, $K'(x, \xi)$ obviously possess the same properties of integrability as the functions $\Phi_j(x)$, $H(x, \xi)$ respectively.

On setting

$$l_j(u) = c_j \qquad (j = 1,2, \cdots ,\nu), \tag{11}$$

$$F(x) + \sum_{j=1}^{\nu} c_j\Phi_j'(x) = \psi(x), \tag{12}$$

we reduce our problem to the Fredholm integral equation

$$u(x) = \psi(x) + \int_a^b K'(x,\xi)u(\xi)d\xi. \tag{13}$$

5. We suppose, in order to include the general case, that the corresponding homogeneous integral equation

$$u(x) = \int_a^b K'(x,\xi)u(\xi)d\xi$$

admits of σ linearly independent solutions

$$\omega_1(x), \quad \omega_2(x), \cdots , \omega_\sigma(x). \tag{14}$$

In this case the associated integral equation

$$v(x) = \int_a^b K'(\xi, x)v(\xi)d\xi$$

has the same number of linearly independent solutions:*

$$\omega_1'(x),\ \omega_2'(x),\ \cdots,\ \omega_\sigma'(x). \tag{15}$$

Either of the sets (14), (15) may be supposed to be orthogonal and normalized. The necessary and sufficient condition that (13) have a solution is given by the system of equations

$$\int_a^b \psi(x)\omega_k'(x)dx = 0 \qquad (k = 1,2,\cdots,\sigma) \tag{16}$$

and if these conditions are satisfied, the most general solution of (13) is

$$u(x) = \psi(x) - \int_a^b \Re'(x,\xi)\psi(\xi)d\xi + \sum_{k=1}^{\sigma} c_k'\omega_k(x) \tag{17}$$

where the c_k' are arbitrary constants, and $\Re'(x, \xi)$ is a so called "pseudo-resolvent kernel" of the kernel $K'(x, \xi)$.

The case where the kernel $K'(x, \xi)$ possesses a reciprocal, may be considered as the special case in which

$$\omega_k(x) = \omega_k'(x) = c_k' = 0 \qquad (k = 1,2,\cdots,\sigma)$$

and $\Re'(x, \xi)$ coincides with this reciprocal.

6. The constants c_j, c_k' remain to be determined. This can be done by substituting (17) into (11) and eliminating $\psi(x)$ from the resulting equations and the relations (12) and (16). Thus we obtain a system of $(\nu+\sigma)$ linear equations for the $(\nu+\sigma)$ unknowns c_j, c_k' as follows:

$$\begin{aligned} \lambda_j(c) &= l_j\left\{ F(x) - \int_a^b \Re'(x,\xi)F(\xi)d\xi \right\} \qquad (j = 1,2,\cdots,\nu), \\ \lambda_k'(c) &= -\int_a^b F(x)\omega_k'(x)dx \qquad (k = 1,2,\cdots,\sigma). \end{aligned} \tag{18}$$

The $\lambda_j(c)$ and $\lambda_k'(c)$ are linear forms in c's and c''s whose coefficients are constants which do not depend upon the function $f(x)$. Denoting by D_0 the determinant of this system, we must consider the two cases

$$D_0 \neq 0\,; \qquad D_0 = 0.$$

* E. W. Hobson, *On the linear integral equation*, Proceedings of the London Mathematical Society, (2), vol. 13 (1914), pp. 307–340. The classical results of the Fredholm theory are extended in this paper to more general kernels, of which ours is a particular case. Cf. also our paper *On Fredholm's integral equations whose kernels are analytic in a parameter*, Annals of Mathematics, (2), vol. 28 (1927), pp. 127–152.

(a) Suppose that $D_0 \neq 0$. In this case system (18) determines uniquely the constants c_j, c_k', and each of these constants is a linear combination of integrals of the form

$$\int_a^b f(t)\beta(t)dt.$$

Substituting these values into (17), we obtain the required expression for $u(x)$:

$$u(x) = \int_a^b \Gamma(x,t)f(t)dt. \tag{III}$$

Conversely, the preceding operations being reversible, it is readily shown that (17) represents a solution of the problem (★). This solution is uniquely determined and by familiar reasoning it is proved that the function $\Gamma(x, t)$ is uniquely determined at all its points of continuity.

Under the following supplementary restrictions:

iii′. The discontinuities of the function $h(x, \xi)$ are regularly distributed, i.e., $h(x, \xi)$ is continuous with respect to ξ for almost all values of x, and continuous in x for almost all values of ξ,*

v′. The operators $l_j(u)$ involve the values of $u(x)$ and its derivatives only at the end points $x=a$, $x=b$,

it can be shown by means of some results of Hobson that $\Gamma(x, t)$ is continuous everywhere on (a, b), except, possibly, along the line $x=t$, where it may have a discontinuity of the first kind.

7. (b) Suppose now that $D_0=0$. We shall prove first that the homogeneous problem (★) has at least one solution which is not identically zero on (a, b). The homogeneous system

$$\lambda_j(c) = 0 \quad (j = 1,2, \cdots ,\nu); \qquad \lambda_k'(c) = 0 \qquad (k = 1,2, \cdots ,\sigma) \tag{19}$$

admits of at least one solution in which not all the constants c_j, c_k' are zero. We substitute these values of c_j, c_k' into (17) and set $f(x)\equiv 0$. The corresponding function $u_0(x)$ is certainly a solution of the homogeneous problem (★), and it only remains to prove that $u_0(x)\not\equiv 0$.

Suppose that $u_0(x)\equiv 0$. Then from (11) it follows that

$$c_j = l_j(u_0) = 0 \qquad (j = 1,2, \cdots ,\nu),$$

and, substituting in (12)

$$c_j = 0, \quad F(x) \equiv 0,$$

* In this case the solution of the problem (★), with continuous $f(x)$ and $\phi_j(x)$, possesses a continuous derivative of order n and satisfies (I′) everywhere.

we have $\psi(x)=0$. Hence

$$\sum_{k=1}^{\sigma} c_k' \omega_k(x) = u_0(x) = 0, \quad c_k' = 0 \quad (k = 1,2, \cdots ,\sigma),$$

contrary to our supposition.

8. We shall prove now that the right-hand members of equations (18) may assume any given set of values if $f(x)$ is suitably chosen. Since this linear system (18) will admit of a solution for the c's only for special values of the right-hand members, it will follow that no solution for a general $f(x)$ is possible and that in this case the Green's function $\Gamma(x, t)$ does not exist.

It follows from the classical theory of the Green's function $G(x, t)$ that, if $F(x)$ is an arbitrary function which possesses a continuous derivative of order n and satisfies the boundary conditions

$$L_i(F) = 0 \qquad (i = 1,2, \cdots ,n), \tag{20}$$

then the function

$$f(x) \equiv L^{(0)}(F)$$

satisfies the relation

$$F(x) = \int_a^b G(x,t)f(t)dt.$$

The question is reduced, therefore, to the proof that there always exists a function $F(x)$ which satisfies conditions (20) and for which the expressions

$$l_j\left\{F(x) - \int_a^b \Re'(x,\xi)F(\xi)d\xi\right\} \qquad (j = 1,2, \cdots ,\nu);$$

$$\int_a^b F(x)\omega_k'(x)dx \qquad (k = 1,2, \cdots, \sigma)$$

take on arbitrary prescribed values. Let $\chi(x)$ be a function subsequently to be determined and determine $F(x)$ from the integral equation

$$\chi(x) = F(x) - \int_a^b \Re'(x,\xi)F(\xi)d\xi.$$

This is always possible, if we use the special form of the pseudo-resolvent kernel $\Re'(x, \xi)$ as given by W. A. Hurwitz.* This kernel $\Re'(x, \xi)$ is defined as the reciprocal of the function

$$K'(x,\xi) - \sum_{k=1}^{\sigma} \omega_k'(x)\omega_k(\xi).$$

* *On the pseudo-resolvent to the kernel of an integral equation*, these Transactions, vol. 13 (1912), pp. 405–418; pp. 405–409.

Hence this latter function is in turn the reciprocal of the kernel $\mathfrak{K}'(x, \xi)$ and we have

$$F(x) = \chi(x) - \int_a^b K'(x,\xi)\chi(\xi)d\xi + \sum_{k=1}^{\sigma} \omega_k'(x) \int_a^b \chi(\xi)\omega_k(\xi)d\xi.$$

On multiplying this by $\omega_s'(x)$ and integrating, we obtain in virtue of the relation

$$\omega_s'(\xi) = \int_a^b K'(x,\xi)\omega_s'(x)dx$$

the result

$$\int_a^b F(x)\omega_s'(x)dx = \int_a^b \chi(\xi)\omega_s(\xi)d\xi \qquad (s = 1,2, \cdots ,\sigma).$$

Now, as functions of x, $G(x, t)$ and therefore $H(x, t)$, $H'(x, t)$ and $K'(x, t)$ satisfy conditions (II), so that

$$L_i(F) = L_i(\chi) + \sum_{k=1}^{\sigma} L_i(\omega_k') \int_a^b \chi(\xi)\omega_k(\xi)d\xi.$$

It is readily seen that, under the conditions (v), § 1, imposed upon the operators $L_i(u)$, $l_j(u)$, the function $\chi(x)$ can always be chosen so that

$$L_i(F) = 0 \qquad (i = 1,2, \cdots ,n)$$

while the expressions

$$l_j(\chi) \qquad (j = 1,2, \cdots ,\nu); \qquad \int_a^b \chi(\xi)\omega_k(\xi)d\xi \qquad (k = 1,2, \cdots ,\sigma)$$

assume the prescribed values. This completes the proof of Theorem 1.

9. THEOREM 2. *Suppose that, under the conditions of Theorem* 1, *the coefficients of the operators of the problem* ($\bigstar$) *depend on a parameter* ρ *and are analytic in an open region* ($\mathfrak{D}$) *of the* ρ*-plane. Suppose that the Green's function* $\Gamma(x, t, \rho)$ *of the problem* ($\bigstar$) *exists for an infinite set of values of* ρ, *which possesses at least one limiting point interior to* ($\mathfrak{D}$). *Then* $\Gamma(x, t, \rho)$ *is a meromorphic function in* ($\mathfrak{D}$). *The homogeneous problem* ($\bigstar$) *is possible when and only when* ρ *is equal to one of the poles of* $\Gamma(x, t, \rho)$ (*characteristic values of the problem* ($\bigstar$)). *For such values of* ρ *the non-homogeneous problem* ($\bigstar$) *is not possible with an arbitrary* $f(x)$, *and the Green's function does not exist.*

To prove this theorem, we shall use another method for obtaining the Green's function $\Gamma(x, t, \rho)$.* We observe that a particular solution of the equation

$$L(u) = \phi(x) \tag{21}$$

is given by

$$Y(x) = \int_a^b g(x,t)\phi(t)dt$$

where

$$g(x,t) = \pm \frac{1}{2\delta(t)} \begin{vmatrix} y_1(x) & , \cdots , y_n(x) \\ y_1^{(n-2)}(t) & , \cdots , y_n^{(n-2)}(t) \\ \cdots & \cdots \\ y_1(t) & , \cdots , y_n(t) \end{vmatrix} \quad \begin{cases} + \text{ if } x > t, \\ - \text{ if } x < t, \end{cases}$$

$$\delta(t) = \begin{vmatrix} y_1^{(n-1)}(t) & , \cdots , y_n^{(n-1)}(t) \\ y_1^{(n-2)}(t) & , \cdots , y_n^{(n-2)}(t) \\ \cdots & \cdots \\ y_1(t) & , \cdots , y_n(t) \end{vmatrix}$$

and $y_1(x), \cdots, y_n(x)$† denotes any fundamental system of solutions of the equation $L(u)=0$.

The general solution of (21) is then

$$u(x) = \sum_{i=1}^{n} c_i'' y_i(x) + Y(x),$$

where $c_1'', \cdots, c_n''$ are arbitrary constants. If we set

$$l_j(u) = c_j \qquad (j = 1,2, \cdots, \mu),$$

$$\psi(x) = \sum_{i=1}^{n} c_i'' y_i(x) + \sum_{j=1}^{\mu} c_j \int_a^b g(x,t)\phi_j(t)dt + \int_a^b g(x,t)f(t)dt,$$

$$H''(x,\xi) = \int_a^b g(x,t)h(t,\xi)dt,$$

we easily obtain from (I) the following integral equation for $u(x)$:

$$u(x) = \psi(x) - \int_a^b H''(x,\xi)\psi(\xi)d\xi.$$

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* For the sake of brevity the letter ρ sometimes will be omitted.

† The fundamental system $y_1(x), \cdots, y_n(x)$ may always be chosen so that $\delta(t)$, which depends on ρ, is $\neq 0$ for all values of ρ.

The kernel $H''(x, \xi)$ is a function of ρ, which is analytic in $(\mathfrak{D})$, and by the classical Fredholm theory it possesses a reciprocal $\mathfrak{H}''(x, \xi)$ which is meromorphic in $(\mathfrak{D})$. Suppose that ρ is not a pole of $\mathfrak{H}''(x, \xi)$. Then we have

$$u(x) = \psi(x) + \int_a^b \mathfrak{H}''(x,\xi)u(\xi)d\xi,$$

and it only remains to determine the constants c_j, c_i'' from the equations

$$L_i(u) = 0 \quad (i = 1,2, \cdots, n); \qquad l_j(u) = c_j \quad (j = 1,2, \cdots, \mu).$$

The determinant $D(\rho)$ of this system is a meromorphic function in $(\mathfrak{D})$, and if it is not identically zero the resulting expression for $\Gamma(x, t, \rho)$ shows that $\Gamma(x, t, \rho)$ is also a meromorphic function in $(\mathfrak{D})$. Thus we obtain the solution of the non-homogeneous problem $(\bigstar)$ for all values of ρ which are different from the poles of either of the functions $\mathfrak{H}''(x, t)$, $\Gamma(x, t, \rho)$ in the form

$$u(x) = \int_a^b \Gamma(x,t,\rho)f(t)dt. \tag{22}$$

Expression (22) represents a solution of the problem $(\bigstar)$:

$$L(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x) + \int_a^b h(x,\xi)u(\xi)d\xi, \tag{I}$$

$$L_i(u) = 0 \qquad (i = 1,2, \cdots, n), \tag{II}$$

even if ρ is a pole of the function $\mathfrak{H}''(x, \xi)$ but not a pole of the function $\Gamma(x, t, \rho)$, for if we exclude from $(\mathfrak{D})$ the immediate vicinities of the poles of $\Gamma(x, t, \rho)$ the left and right-hand members of equations (I), (II) are analytic throughout the remaining region. Since they have been proved to be equal in a part of this region it follows that they must be equal throughout the whole of it. Now, under the conditions of Theorem 2, we can prove that the determinant $D(\rho)$ cannot be identically zero in $(\mathfrak{D})$, for if it were, the homogeneous problem $(\bigstar)$ would have a solution for all values of ρ which are not poles of $\mathfrak{H}''(x, \xi)$. By virtue of Theorem 1, then the Green's function of the problem $(\bigstar)$ would not exist, except possibly for ρ one of the excepted values. Since this set of values has no limiting point in the interior of $(\mathfrak{D})$ the conditions of Theorem 2 would be contradicted. This completes the proof of Theorem 2.*

* There are examples where the Green's function $\Gamma(x, t, \rho)$ exists for no values of ρ, for instance the system $u' + \rho u = f(x) + \rho \int_0^1 u dx$; $u(1) = u(0)$. In this connection it would be of interest to determine whether the requirement of Theorem 2 concerning the set of points of existence of $\Gamma(x, t, \rho)$ might be replaced by a less restrictive one, as for instance that the Green's function $\Gamma(x, t, \rho)$ exist at a finite number of points, or, even, at a single point of the region $(\mathfrak{D})$. This can be done in the case of a differential problem.

II. The Green's function of the differential problem

10. We shall specialize our problem (★) now by introducing certain restrictions upon the coefficients of the operators involved. Under these restrictions, which will be enunciated as they become necessary, we have at hand established facts from the theory of differential equations. With the use of these we shall derive formulas which reveal the structural features of the Green's function $G(x, t, \rho)$. Thus we make the following assumptions:

(A) i. The operators of the differential problem (★) are polynomials in ρ, i.e. our problem is of the form

$$L(u) \equiv u^{(n)} + P_1(x,\rho)u^{(n-1)} + \cdots + P_n(x,\rho)u,$$

$$L_i(u) \equiv \sum_{s=0}^{n} \rho^s L_i^{(s)}(u) \equiv A_i(u,\rho) + B_i(u,\rho) + \int_a^b \alpha_i(x,\rho)u(x)dx \qquad (i = 1,2,\cdots,n),$$

where

$$P_i(x,\rho) \equiv \rho^i \sum_{j=0}^{i} \rho^{-j} p_{ij}(x),$$

$$L_i^{(s)}(u) \equiv A_i^{(s)}(u) + B_i^{(s)}(u) + \int_a^b \alpha_{is}(x)u(x)dx,$$

$$A_i^{(s)}(u) \equiv \sum_{k=1}^{n} a_{iks}u^{(k-1)}(a)\ ; \quad B_i^{(s)}(u) \equiv \sum_{k=1}^{n} b_{iks}u^{(k-1)}(b),$$

$$A_i(u,\rho) \equiv \sum_{s=0}^{n} \rho^s A_i^{(s)}(u)\ ; \quad B_i(u,\rho) \equiv \sum_{s=0}^{n} \rho^s B_i^{(s)}(u),$$

$$\alpha_i(x,\rho) \equiv \sum_{s=0}^{n} \rho^s \alpha_{is}(x).$$

ii. The functions

$$p_{ij}(x) \qquad (i = 1,2,\cdots,n; j = 0,1,\cdots,i)$$

are of bounded variation on (a, b) and possess continuous derivatives of order $(n-i)$. The functions

$$\frac{d^2 p_{i0}(x)}{dx^2}, \qquad \frac{dp_{i1}(x)}{dx}$$

are of bounded variation on (a, b).

iii. The "characteristic" equation

$$\mathfrak{F}(\theta) \equiv \theta^n + p_{10}(x)\theta^{n-1} + \cdots + p_{n0}(x) = 0$$

has simple roots for all values of x in (a, b). These roots

$$\theta_1(x), \cdots, \theta_n(x)$$

are of the form

$$\theta_i(x) = \pi_i q_i(x) \quad (\pi_i \text{ constant}; \ i = 1,2, \cdots, n) \tag{1}$$

where either

$$\pi_i = \pm \pi_0 \quad (\pi_0 \text{ real or complex and } \neq 0) \text{ and } q_i(x) \geqq q_0 > 0 \text{ on } (a,b),$$

or else

$$\pi_i \neq 0 \text{ and } q_1(x) \equiv q_2(x) \equiv \cdots \equiv q_n(x) \equiv q(x) \geqq q_0 > 0 \text{ on } (a,b).$$

iv. The operators

$$A_i(u,\rho) + B_i(u,\rho) \qquad (i = 1,2, \cdots, n)$$

are linearly independent for all values of ρ.

v. The functions

$$\alpha_{is}(x)$$

possess continuous derivatives of the first order which are of bounded variation on (a, b).*

11. It is well known under the conditions stated that the complex ρ-plane may be divided into a finite number of sectors $(\Re)$, such that, in each sector the numbers

$$w_i = \int_a^b \theta_i(x)dx = \pi_i \int_a^b q_i(x)dx \qquad (i = 1,2, \cdots, n) \tag{2}$$

may be ordered as follows:†

$$\operatorname{Re}\rho w_1 \leqq \cdots \leqq \operatorname{Re}\rho w_r \leqq 0 \leqq \operatorname{Re}\rho w_{r+1} \leqq \cdots \leqq \operatorname{Re}\rho w_n \text{ in } \Re, \tag{3}$$

and further that there exists a fundamental system of solutions of the equation $L(y) = 0$, which, in any $\Re$, is of the form‡

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* The integral terms of the boundary operators used in D(17–19, 26) are $\int_a^b \alpha_i(x, \rho)u^{(n-1)}(x)dx$ while those here are $\int_a^b \alpha_i(x, \rho)\, u(x)\, dx$. This explains a slight difference between the following results and those of D.

† These sectors may always be so constructed that none of them contain more than one of the rays $\operatorname{Re}\rho w_i = \operatorname{Re}\rho w_k$, and none of these rays serve as a boundary.

‡ The symbol E, $E(\rho)$, $E(\rho, \cdots)$ is used as a generic notation to designate functions of ρ (and other variables), which remain bounded for large $|\rho|$. The symbol ϵ, $\epsilon(\rho)$, $\epsilon(\rho, \cdots)$ will be used to designate functions of ρ (and other variables) which tend uniformly to zero as ρ becomes infinite.

$$y_i(x,\rho) = e^{\rho\int_a^x \theta_i(x)dx}\left\{\eta_i(x) + \frac{1}{\rho}\eta_{i0}(x) + \frac{E}{\rho^2}\right\} = e^{\rho\int_a^x \theta_i(x)dx}\left\{\eta_i(x) + \frac{E}{\rho}\right\},$$

(4)

$$\frac{d^s y_i(x,\rho)}{dx^s} = e^{\rho\int_a^x \theta_i(x)dx}\rho^s\{\theta_i(x)\}^s\left\{\eta_i(x) + \frac{1}{\rho}\eta_{is}(x) + \frac{E}{\rho^2}\right\}$$

$$(i = 1,2,\cdots,n\,;\ s = 0,1,\cdots,n-1).$$

The functions*

$$\eta_i(x)$$

possess derivatives of the second order, and the functions

$$\eta_{is}(x)$$

possess derivatives of the first order which are continuous and of bounded variation on (a, b).

Lastly it is known that if $Y_k(t,\rho)$ denotes the cofactor of $y_k^{(n-1)}(t,\rho)$ in the determinant

$$\delta(t,\rho) \equiv \begin{vmatrix} y_1^{(n-1)}(t,\rho) & \cdots & y_n^{(n-1)}(t,\rho) \\ \cdots & \cdots & \cdots \\ y_1(t,\rho) & \cdots & y_n(t,\rho) \end{vmatrix},$$

then the adjoint equation $L'(y)=0$ admits of a fundamental system of solutions

$$z_k(t,\rho) = \frac{Y_k(t,\rho)}{\delta(t,\rho)} \qquad (k = 1,2,\cdots,n),$$

and in any $\mathfrak{R}$

$$z_k(t,\rho) = e^{-\rho\int_a^t \theta_k(x)dx}\rho^{-n+1}\left\{\zeta_k(t) + \frac{1}{\rho}\zeta_{k0}(t) + \frac{E}{\rho^2}\right\},$$

(5)

$$\frac{d^s z_k(t,\rho)}{dt^s} = e^{-\rho\int_a^t \theta_k(x)dx}\rho^{-n+1+s}\left\{\zeta_k(t) + \frac{1}{\rho}\zeta_{ks}(t) + \frac{E}{\rho^2}\right\}(-1)^s\{\theta_k(t)\}^s$$

$$(k = 1,2,\cdots,n\,;\ s = 0,1,\cdots,n-1),$$

where

$$\zeta_k(t) = (\eta_k(t)\mathfrak{F}'\{\theta_k(t)\})^{-1}$$

---

* It can be proved (D, 2) that

$$\eta_k(x) = \frac{1}{(\mathfrak{F}'(\theta_i))^{1/2}}\exp\left\{-\int_a^x (\mathfrak{F}_1(\theta_i)/\mathfrak{F}'(\theta_i))dx\right\}$$

where

$$\mathfrak{F}'(\theta) \equiv \frac{d\mathfrak{F}(\theta)}{d(\theta)}\,;\ \mathfrak{F}_1(\theta) \equiv p_{11}(x)\theta^{n-1} + \cdots + p_{n-1,1}(x)\theta + p_{n1}(x).$$

and where $\zeta_k(t)$, $\zeta_{ks}(t)$ possess the same properties of continuity as $\eta_i(x)$, $\eta_{is}(x)$ respectively.

12. The Green's function of the differential problem (★) is given by

$$G(x,t,\rho) = \frac{(-1)^n\Delta(x,t,\rho)}{\Delta(\rho)}, \tag{6}$$

where

$$\Delta(\rho) = \begin{vmatrix} u_{11}(\rho) & \cdots & u_{1n}(\rho) \\ \cdots & \cdots & \cdots \\ u_{n1}(\rho) & \cdots & u_{nn}(\rho) \end{vmatrix}, \tag{7}$$

$$u_{ik}(\rho) = L_i(y_k), \tag{8}$$

$$\Delta(x,t,\rho) = \begin{vmatrix} y_1(x,\rho) & \cdots & y_n(x,\rho) & g_0(x,t,\rho) \\ u_{11}(\rho) & \cdots & u_{1n}(\rho) & g_1(t,\rho) \\ \cdots & \cdots & \cdots & \cdots \\ u_{n1}(\rho) & \cdots & u_{nn}(\rho) & g_n(t,\rho) \end{vmatrix}, \tag{9}$$

$$g_0(x,t,\rho) = \begin{cases} \sum_{k=1}^{r} y_k(x,\rho)z_k(t,\rho) \text{ if } x > t, \\ -\sum_{k=r+1}^{n} y_k(x,\rho)z_k(t,\rho) \text{ if } x < t, \end{cases} \tag{10}$$

$$\begin{aligned} g_i(t,\rho) = &-\sum_{k=r+1}^{n} A_i(y_k)z_k(t,\rho) + \sum_{k=1}^{r} B_i(y_k)z_k(t,\rho) \\ &-\sum_{k=r+1}^{n} z_k(t,\rho)\int_a^t \alpha_i(x,\rho)y_k(x,\rho)dx \\ &+\sum_{k=1}^{r} z_k(t,\rho)\int_t^b \alpha_i(x,\rho)y_k(x,\rho)dx. \end{aligned} \tag{11}$$

Let l_{is} denote the greatest value of the index l for which, with i and s fixed, at least one of the coefficients a_{isl}, b_{isl} is different from zero, so that

$$a_{isl} = 0, \quad b_{isl} = 0 \text{ if } l > l_{is},$$
$$|a_{isl}| + |b_{isl}| > 0 \text{ if } l = l_{is}.$$

If

$$a_{isl} = b_{isl} = 0 \text{ for } l = 1,2,\cdots,n,$$

we set $l_{is}=0$.

Also let l_i be the greatest number of the set

$$l_{is} + s - 1 \qquad (s = 0,1,\cdots,n)$$

with $l_{is}>0$. Then obviously

$$A_i(y_k) = \rho^{l_i}[A_{ik}]\ ;\ B_i(y_k) = \rho^{l_i}e^{\rho w_k}[B_{ik}]^* \tag{12}$$

where

$$A_{ik} = \eta_k(a)\sum_{(s)} a_{isl_s}\{\theta_k(a)\}^{l_{is}-1},$$
$$B_{ik} = \eta_k(b)\sum_{(s)} b_{isl_s}\{\theta_k(b)\}^{l_{is}-1}, \qquad l = l_{is}, \tag{13}$$

and the summation is taken over all values of s for which $l_{is}+s-1=l_i$. We have further

$$\int_a^b \alpha_{is}(x)y_k(x,\rho)dx = \frac{1}{\rho}\{[-\alpha_{is}(a)\eta_k(a)\theta_k(a)^{-1}] + e^{\rho w_k}[\alpha_{is}(b)\eta_k(b)\theta_k(b)^{-1}]\} \tag{14}$$

which may be easily proved by integration by parts, if we take into account condition (**A**, v) and the fact that if c is any constant different from zero, and $\psi(z)$ is any function of bounded variation on $(0, Z)$, then

$$\int_0^Z e^{cpz}\psi(z)dz = \frac{E}{\rho} + \frac{E}{\rho}e^{cpZ}. \tag{15}$$

In the same way we obtain

$$\int_a^t \alpha_{is}(x)y_k(x,\rho)dx = \frac{1}{\rho}\{[-\alpha_{is}(a)\eta_k(a)\theta_k(a)^{-1}] + e^{\rho\int_a^t \theta_k(x)dx}[\alpha_{is}(t)\eta_k(t)\theta_k(t)^{-1}]\}, \tag{16}$$

$$\int_t^b \alpha_{is}(x)y_k(x,\rho)dx = \frac{1}{\rho}e^{\rho w_k}\{[\alpha_{is}(b)\eta_k(b)\theta_k(b)^{-1}] - e^{\rho\int_a^t \theta_k(x)dx}[\alpha_{is}(t)\eta_k(t)\theta_k(t)^{-1}]\}.$$

13. We shall now make the following assumptions:

(**B**) i. If s_i denotes the greatest value of the index s for which $\alpha_{is}(x)\not\equiv 0$, so that

$$\alpha_{is}(x) \equiv 0 \text{ if } s > s_i\ ;\ \alpha_{is}(x) \equiv \alpha^{(i)}(x) \not\equiv 0 \text{ on } (a, b) \text{ if } s = s_i,$$

then

$$s_i < l_i + 1 \qquad (i = 1, 2, \cdots, n).$$

Under this condition we find immediately that

$$u_{ik}(\rho) = \rho^{l_i}\{[A_{ik}] + e^{\rho w_k}[B_{ik}]\}. \tag{17}$$

---

* The notation $[\phi]$ is due to G. D. Birkhoff and designates the expression of the form $\phi+E/\rho$.

Hence from (7)

$$\Delta(\rho) = \rho^{l} \begin{vmatrix} [A_{11}] + e^{\rho w_1}[B_{11}] & \cdots & [A_{1n}] + e^{\rho w_n}[B_{1n}] \\ \cdots & \cdots & \cdots \\ [A_{n1}] + e^{\rho w_1}[B_{n1}] & \cdots & [A_{nn}] + e^{\rho w_n}[B_{nn}] \end{vmatrix} \tag{18}$$

where

$$l = l_1 + l_2 + \cdots + l_n$$

and, if we denote by $\Delta_{ik}(\rho)$ the cofactor of $u_{ik}(\rho)$ in the determinant $\Delta(\rho)$ we have

$$\Delta_{ji}(\rho) = \begin{cases} \rho^{l-l_i} e^{\rho w} E_{ji}(\rho) & \text{if } i = 1, 2, \cdots, \tau, \\ \rho^{l-l_i} e^{\rho(w-w_i)} E_{ji}(\rho) & \text{if } i = \tau+1, \cdots, n, \end{cases} \tag{19}$$

where

$$w = w_{\tau+1} + w_{\tau+2} + \cdots + w_n.$$

For the sake of brevity we set

$$x_i = \int_a^x q_i(x)dx\ ;\ \xi_i = \int_a^t q_i(x)dx\ ;\ X_i = \int_a^b q_i(x)dx\ ;\ w_i = \pi_i X_i. \tag{20}$$

A simple computation gives then

$$g_0(x,t,\rho) = \begin{cases} \rho^{-n+1} \sum_{k=1}^{\tau} e^{\rho\pi_k(x_k-\xi_k)} \left\{ \eta_k(x)\zeta_k(t) + \dfrac{H_k(x,t)}{\rho} + \dfrac{E}{\rho^2} \right\} \text{if } x > t, \\ -\rho^{-n+1} \sum_{k=\tau+1}^{n} e^{\rho\pi_k(x_k-\xi_k)} \left\{ \eta_k(x)\zeta_k(t) + \dfrac{H_k(x,t)}{\rho} + \dfrac{E}{\rho^2} \right\} \text{if } x < t, \end{cases} \tag{21}$$

where the functions $H_k(x, t)$ are polynomials in

$$\eta_i(x),\ \eta_{i0}(x),\ \zeta_i(t),\ \zeta_{i0}(t),$$

and, a fortiori, possess derivatives of the first order which are continuous and of bounded variation in each of the variables x and t, the total variation being uniformly bounded on (a, b).

Also

$$g_i(t,\rho) = \rho^{l_i-n+1} \left\{ -\sum_{k=\tau+1}^{n} e^{-\rho\pi_k\xi_k}[A_{ik}\zeta_k(t)] + \sum_{k=1}^{\tau} e^{\rho\pi_k(X_k-\xi_k)}[B_{ik}\zeta_k(t)] + \frac{1}{\rho}\left[\frac{a_i(t)}{p_{n0}(t)}\right] \right\} \tag{22}$$

---

* In deducing (22) from (11), (12), (16) we use the relations which are satisfied by the roots $z_1, z_2, \cdots, z_n$ of any algebraic equation

$$P(z) \equiv z^n + A_1 z^{n-1} + \cdots + A_{n-1} z + A_n = 0,$$

namely

$$\sum_{i=1}^{n} (z_i)^s / (P'(z_i)) = \begin{cases} 0 \text{ if } s = 0, 1, \cdots, n-2, \\ 1 \text{ if } s = n-1, \\ -1/A_n \text{ if } s = -1. \end{cases}$$

where

$$a_i(t) = \begin{cases} \alpha^{(i)}(t) \text{ if } s_i = l_i \\ 0 \quad \text{if } s_i < l_i. \end{cases} \tag{23}$$

A further restriction now is the following:

(**B**) ii. If σ is the largest integer for which Re $\rho w_\sigma < 0$ throughout $(\Re)$ then all the constants

$$M_1 = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1\sigma} & B_{1\sigma+1} & \cdots & B_{1n} \\ A_{21} & A_{22} & \cdots & A_{2\sigma} & B_{2\sigma+1} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{n\sigma} & B_{n\sigma+1} & \cdots & B_{nn} \end{vmatrix},$$

$$M_2 = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1\tau} & B_{1\tau+1} & \cdots & B_{1n} \\ A_{21} & A_{22} & \cdots & A_{2\tau} & B_{2\tau+1} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{n\tau} & B_{n\tau+1} & \cdots & B_{nn} \end{vmatrix}$$

are to be different from zero.*

Under the conditions (**A**), (**B**) there exist infinitely many poles of the Green's function $G(x, t, \rho)$ which are the characteristic values of the differential problem (★) (D, 20–26). Moreover, if the interiors of small circles with a common radius δ around these roots are all excluded from the ρ-plane, then in the remaining part $(\Re_\delta)$ of the sector $(\Re)$

$$|\Delta(\rho)e^{-\rho w}\rho^{-l}| \geqq N_\delta > 0 \tag{24}$$

where N_δ is a positive constant which does not depend on ρ, and may be taken the same for all the sectors $(\Re)$.

14. These results we shall apply now to the discussion of the behavior of the Green's function $G(x, t, \rho)$ for large values of ρ in $(\Re_\delta)$. We have from (6), (7), and (9)

$$G(x,t,\rho) = g_0(x,\tau,\rho) + \sum_{i,j=1}^{n} \frac{\Delta_{ji}(\rho)y_i(x,\rho)g_j(t,\rho)}{\Delta(\rho)}. \tag{25}$$

Hence if we set

$$\begin{aligned} &e^{\rho\pi_i x_i} = \varpi_i' \ (i = 1,2,\cdots,\tau)\ ;\ e^{-\rho\pi_i(X_i - x_i)} = \varpi_i' \quad (i = \tau+1,\cdots,n), \\ &e^{\rho\pi_i(X_i - \xi_i)} = \varpi_i'' \ (i = 1,2,\cdots,\tau)\ ;\ e^{-\rho\pi_i\xi_i} = \varpi_i'' \quad (i = \tau+1,\cdots,n), \end{aligned} \tag{26}$$

* It should be noted that the constants τ, σ, A_{ik}, B_{ik} and hence M_1 and M_2 depend on the sector $(\Re)$.

we obtain

$$G(x,t,\rho) = g_0(x,t,\rho) + \rho^{-n+1}Q_0(x,t,\rho) + \rho^{-n}R_0(x,t,\rho). \tag{27}$$

Here $Q_0(x, t, \rho)$ is a bilinear form in two sets of quantities ϖ_i' and ϖ_i''; i.e.,

$$Q_0(x,t,\rho) = \sum_{i,k=1}^{n} Q_{ik0}(x,t,\rho)\varpi_i'\varpi_k'', \tag{28}$$

the coefficients Q_{ik0} being of the form

$$Q_{ik0}(x,t,\rho) = [\omega_{ik0}(x,t)]E_{ik}(\rho). \tag{29}$$

The quantities $E_{ik}(\rho)$ depend only on ρ and remain bounded in $(\mathfrak{R}_\delta)$, while the functions $\omega_{ik0}(x, t)$ depend only on x and t, each being a sum of products of a function of x by a function of t whose second derivatives are continuous on (a, b).

$R_0(x, t, \rho)$ is a bilinear form in ϖ_i' and $a_i(t)$:

$$R_0(x,t,\rho) = \sum_{i,k=1}^{n} R_{ik0}(x,t,\rho)\varpi_i a_k(t), \tag{30}$$

$$R_{ik0}(x,t,\rho) = [r_{i0}(x)/p_{n0}(t)]E_{ik}(\rho), \tag{31}$$*

where the functions $r_{i0}(x)$ depend only on x and possess second derivatives which are continuous on (a, b).

If $s_i < l_i (i = 1, 2, \cdots, n)$, all the functions $a_i(t)$ vanish and the form $R_0(x, t, \rho)$ reduces to zero.

In exactly the same fashion it can be proved that

$$\frac{\partial^s G(x,t,\rho)}{\partial x^s} = g_{0s}(x,t,\rho) + \rho^{-n+s+1}Q_s(x,t,\rho) + \rho^{-n+s}R_s(x,t,\rho) \tag{32}$$

where

$$g_{0s}(x,t,\rho) \equiv \frac{\partial^s g_0(x,t,\rho)}{\partial x^s}$$

$$= \begin{cases} \rho^{-n+s+1}\displaystyle\sum_{k=1}^{r} e^{\rho\pi_k(x_k-\xi_k)}\left\{(\theta_k(x))^s\eta_k(x)\zeta_k(t) + \frac{H_{ks}(x,t)}{\rho} + \frac{E}{\rho^2}\right\}, \\ -\rho^{-n+s+1}\displaystyle\sum_{k=r+1}^{n} e^{\rho\pi_k(x_k-\xi_k)}\left\{(\theta_k(x))^s\eta_k(x)\zeta_k(t) + \frac{H_{ks}(x,t)}{\rho} + \frac{E}{\rho^2}\right\}, \end{cases} \tag{33}$$

according as $x > t$ or $x < t$;

* The coefficients $E_{ik}(\rho)$ are of course not the same as in (29).

$$(34)\quad Q_s(x,t,\rho) = \sum_{i,k=1}^{n} Q_{iks}(x,t,\rho)\varpi_i'\varpi_k''\,;\quad Q_{iks}(x,t,\rho) = [\omega_{iks}(x,t)]E_{ik}(\rho),$$

$$(35)\quad R_s(x,t,\rho) = \sum_{i,k=1}^{n} R_{iks}(x,t,\rho)\varpi_i' a_k(t)\,;\quad R_{iks}(x,t,\rho) = [r_{is}(x)/p_{n0}(t)]E_{ik}(\rho),$$

and $\omega_{iks}(x, t)$, $r_{is}(x)$ are functions analogous to $\omega_{ik0}(x, t)$, $r_{i0}(x)$.

15. Let us suppose now that conditions (**A**), (**B**) are satisfied and set

$$(36)\quad T_1(f,x) \equiv \rho^{n-1}\int_a^b G(x,t,\rho)p_{n0}(t)f(t)dt,$$

$$(37)\quad T_2(f,t) \equiv \rho^{n-1}\int_a^b G(x,t,\rho)p_{n0}(x)f(x)dx.$$

Further let us denote by $(\mathfrak{D}_\delta)$ that part of the ρ-plane which is outside (or on the boundaries) of the circles with a common small radius δ around the poles of the Green's function $G(x, t, \rho)$. Then the representation (27) of the Green's function $G(x, t, \rho)$ enables us to prove the following

LEMMA 1. (1) *If $f(x)$ is any function integrable on (a, b), then the integrals* (36) *and* (37) *tend to zero uniformly on (a, b), as ρ becomes infinite, remaining in $(\mathfrak{D}_\delta)$. The same is true of the product*

$$\rho^{-s}\frac{d^s T_1(f,x)}{dx^s} \qquad (s = 1,2,\cdots,n-1).$$

(2) *If $f(x)$ is any function of bounded variation on (a, b) we have in $(\mathfrak{D}_\delta)$*

$$(38)\quad |T_1(f,x)| < \frac{NV_f}{|\rho|}\,;\qquad |T_2(f,t)| < \frac{NV_f}{|\rho|}$$

where N denotes a positive constant which depends only on δ (and on the coefficients of the problem $(\bigstar)$) and V_f is the greater of the two numbers, namely the upper limit of $f(x)$ and the total variation of $f(x)$ on (a, b). The same is true of the product

$$\rho^{-s}\frac{d^s T_1(f,x)}{dx^s} \qquad (s = 1,2,\cdots,n-1).$$

(3) *If $f(x)$ is any function which possesses a first derivative of bounded variation on (a, b), we have in $(\mathfrak{D}_\delta)$*

$$(39)\quad T_1(f,x) = \frac{1}{\rho}f(x) + \frac{1}{\rho}\Lambda_1(\varpi_i') + \frac{E}{\rho^2},$$

$$(40)\quad T_2(f,t) = \frac{1}{\rho}f(t) + \frac{1}{\rho}\Lambda_2(\varpi_i'') + \frac{E}{\rho^2},$$

where $\Lambda_1(\varpi_i')$, $\Lambda_2(\varpi_i'')$ *are linear forms in* ϖ_i', ϖ_i'' *respectively:**

$$\Lambda_1(\varpi_i') = \sum_{i=1}^{n} [\omega_{i1}(x)]\varpi_i' E_i(\rho),$$

$$\Lambda_2(\varpi_i'') = \sum_{i=1}^{n} [\omega_{i2}(x)]\varpi_i'' E_i(\rho),$$

the coefficients $\omega_{i1}(x)$, $\omega_{i2}(x)$ *being continuous and of bounded variation on* (a, b).

16. The proof is based upon the following two simple lemmas:

LEMMA 2. *If* $\psi(t)$ *is any function integrable on* (a_0, b_0), *and* c *is any constant* $\neq 0$, *then the integral*

$$\int_\alpha^\beta e^{c\rho t}\psi(t)dt \qquad (0 \leqq a_0 \leqq \alpha \leqq \beta \leqq b_0)$$

tends uniformly to zero, as $\rho \to \infty$ *remaining in the half-plane* $\operatorname{Re} c\rho \leqq 0$ (D, 38).

LEMMA 3. *If* $\psi(t)$ *is any function of bounded variation on* (a_0, b_0) *and* c *is any constant* $\neq 0$, *then in the half-plane* $\operatorname{Re} c\rho \leqq 0$,

$$\int_\alpha^\beta e^{c\rho t}\psi(t)dt = \frac{V_\psi}{\rho}E \qquad (0 \leqq a_0 \leqq \alpha \leqq \beta \leqq b_0),$$

where E *is a bounded function whose upper bound does not depend on* ψ, α *or* β.

The proof of Lemma 3 follows immediately from the second Law of the Mean, if we represent the function $\psi(t)$ (or its real and imaginary parts in case it is complex) as a difference of two monotonic functions.

17. To prove Lemma 1, we use (27) and (32). These give

$$(41) \quad T_1(f,x) = \rho^{n-1}\int_a^b g_0(x,t,\rho)p_{n0}(t)f(t)dt + \int_a^b Q_0(x,t,\rho)p_{n0}(t)f(t)dt + \frac{1}{\rho}\int_a^b R_0(x,t,\rho)p_{n0}(t)f(t)dt,$$

$$(42) \quad \rho^{-s}\frac{d^s T_1(f,x)}{dx^s} = \rho^{n-s-1}\int_a^b g_{0s}(x,t,\rho)p_{n0}(t)f(t)dt + \int_a^b Q_s(x,t,\rho)p_{n0}(t)f(t)dt + \frac{1}{\rho}\int_a^b R_s(x,t,\rho)p_{n0}(t)f(t)dt.$$

* In the following, the symbol Λ will be used to designate any linear form in $\bar{\omega}_i'$ or $\bar{\omega}_i''$, of the type Λ_1 or Λ_2.

The statement (1) of Lemma 1 follows immediately from Lemma 2 and the statement (2) follows from Lemma 3. In order to prove the statement (3), we observe first that, by virtue of Lemma 3, the sum of the second and the third terms in (41) is of the form

$$\frac{1}{\rho}\Lambda(\varpi_i') + \frac{E}{\rho^2}.$$

On substituting (21) into the first term of (41) we can rewrite it in the form

$$\begin{aligned}
&\sum_{k=1}^{r}\int_a^x e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\zeta_k(t)p_{n0}(t)f(t)dt \\
&-\sum_{k=r+1}^{n}\int_x^b e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\zeta_k(t)p_{n0}(t)f(t)dt \\
&+\frac{1}{\rho}\sum_{k=1}^{r}\int_a^x e^{\rho\pi_k(x_k-\xi_k)}\mathrm{H}_k(x,t)p_{n0}(t)f(t)dt \\
&-\frac{1}{\rho}\sum_{k=r+1}^{n}\int_a^b e^{\rho\pi_k(x_k-\xi_k)}\mathrm{H}_k(x,t)p_{n0}(t)f(t)dt + \frac{E}{\rho^2}.
\end{aligned}$$

The application of Lemma 3 shows that the coefficient of $1/\rho$ here is also E/ρ.

An integration by parts in the two first terms yields

$$\begin{aligned}
&-\frac{1}{\rho}\sum_{k=1}^{n}\frac{\eta_k(x)\zeta_k'(x)p_{n0}(x)f(x)}{\theta_k(x)} + \frac{1}{\rho}\Lambda(\varpi_i') \\
&+\frac{1}{\rho}\sum_{k=1}^{r}\int_a^x e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\frac{d}{dt}\frac{\zeta_k(t)p_{n0}(t)f(t)}{\theta_k(t)}dt \\
&-\frac{1}{\rho}\sum_{k=r+1}^{n}\int_x^b e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\frac{d}{dt}\frac{\zeta_k(t)p_{n0}(t)f(t)}{\theta_k(t)}dt,
\end{aligned}$$

which by virtue of the footnote on page 772 and Lemma 3 is equal to

$$\frac{1}{\rho}f(x) + \frac{1}{\rho}\Lambda(\varpi_i') + \frac{E}{\rho^2}.$$

On collecting this material we have the complete proof of (36). An analogous proof may be carried through for (37).

LEMMA 4. *If $f_1(x)$ and $f_2(x)$ are any two functions whose first derivatives are of bounded variation on (a, b), then under the conditions of Lemma* 1,

$$\rho^n \int_a^b \int_a^b G(x,t,\rho) f_1(x) f_2(t) p_{n0}(t) dx dt = \left[\int_a^b f_1(x) f_2(x) dx \right], \tag{43}$$

$$\rho^n \int_a^b \int_a^b G(x,t,\rho) f_1(t) f_2(x) p_{n0}(x) dx dt = \left[\int_a^b f_1(x) f_2(x) dx \right]. \tag{44}$$

The proof is given by (39) if we observe that, by virtue of Lemma 3,

$$\int_a^b \Lambda_1(\varpi_i') f_1(x) dx = \frac{E}{\rho} \text{ on } (\mathfrak{D}_\delta).$$

18. **Remark 1.** Lemmas 1 and 4 hold true if the functions $f(x), f_1(x), f_2(x)$ depend on any number of other variables, provided that they are uniformly bounded or else that their total variation or that of their derivatives is uniformly bounded.

Remark 2. It follows immediately from (27) and (32) that, in $(\mathfrak{D}_\delta)$,

$$|G(x,t,\rho)| < N|\rho|^{-n+1}; \quad \left| \frac{\partial^s G(x,t,\rho)}{\partial x^s} \right| < N|\rho|^{-n+1+s}$$

where N may be taken the same as in (38).

III. The equiconvergence theorem

19. We return now to the general problem

$$L(u) = f(x) + \sum_{j=1}^{\mu} l_j(u) \phi_j(x,\rho) + \int_a^b h(x,\xi,\rho) u(\xi) d\xi, \tag{I}$$

$$L_i(u) = 0 \qquad (i = 1,2,\cdots,n). \tag{II}$$

In addition to the conditions (**A**), (**B**) we assume also the following:

(**C**) i. The functions

$$h(x,\xi,\rho), \ \phi_j(x,\rho) \qquad (j = 1,2,\cdots,\mu)$$

are polynomials in ρ, of at most the degree $(n-1)$.

ii. The operators

$$l_j(u) \qquad (j = 1,2,\cdots,n)$$

do not depend on ρ.

iii. If m_j denotes the degree of $\phi_j(x,\rho)$ and λ_j denotes the order of the highest derivative which occurs in the operator $l_j(u)$, then

$$\lambda_k + m_j \leqq n-1 \qquad (k,j = 1,2,\cdots,\mu). \tag{1}$$

iv. If $\lambda_k + m_j = n-1$, then the coefficient of the highest power of ρ in $\phi_j(x,\rho)$ is of bounded variation on (a,b). The coefficient of ρ^{n-1} in $h(x,\xi,\rho)$

is of bounded variation in ξ for each value of x, the total variation being uniformly bounded on (a, b). The remaining coefficients in $h(x, \xi, \rho)$, $\phi_j(x, \rho)$ are integrable on (a, b), with respect either to x, or to ξ, or with respect to x and ξ.

v. The operators

$$l_j(u),\ A_i(u) + B_i(u) \qquad (j = 1,2, \cdots ,\mu\ ;\ i = 1,2, \cdots ,n)$$

are linearly independent for all values of ρ.

20. We see at once that the conditions of Theorems 1 and 2 are fulfilled in this case, and now we proceed to the explicit expression of the Green's function $\Gamma(x, t, \rho)$ of the problem $(\bigstar)$.

Using the method indicated in §§4–6 we set

$$F(x,\rho) \equiv \int_a^b G(x,t,\rho)f(t)dt, \tag{2}$$

$$\Phi_j(x,\rho) \equiv \int_a^b G(x,t,\rho)\phi_j(t,\rho)dt, \tag{3}$$

$$H(x,\xi,\rho) \equiv \int_a^b G(x,t,\rho)h(t,\xi,\rho)dt, \tag{4}$$

$$l_j(u) = c_j \qquad (j = 1,2, \cdots ,\mu), \tag{5}$$

$$\Psi(x,\rho) \equiv \sum_{j=1}^{\mu} c_j \Phi_j(x,\rho) + F(x,\rho). \tag{6}$$

Suppose for a moment that ρ is not a pole of $G(x, t, \rho)$, nor of $\mathfrak{H}(x, \xi, \rho)$, the reciprocal of the kernel $H(x, \xi, \rho)$. The problem $(\bigstar)$ is equivalent to the integral equation

$$u(x,\rho) = \Psi(x,\rho) + \int_a^b H(x,\xi,\rho)u(\xi,\rho)d\xi, \tag{7}$$

together with the conditions (5). Hence

$$u(x,\rho) = \Psi(x,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)\Psi(\xi,\rho)d\xi = \Omega(x,\rho) + \sum_{j=1}^{\mu} c_j\Omega_j(x,\rho), \tag{8}$$

where

$$\Omega(x,\rho) \equiv F(x,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)F(\xi,\rho)d\xi, \tag{9}$$

$$\Omega_j(x,\rho) \equiv \Phi_j(x,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)\,\Phi_j(\xi,\rho)d\xi. \tag{10}$$

If we set

$$\chi_{ik}(\rho) \equiv \begin{cases} -\ l_i(\Omega_k) \text{ if } k \neq i\ ; \\ 1 - l_i(\Omega_i) \text{ if } k = i\ ; \end{cases} \quad \chi_i(\rho) \equiv l_i(\Omega) \qquad (i,k = 1,2,\cdots,\mu), \tag{11}$$

equations (5) can be written as follows:

$$\sum_{k=1}^{\mu} c_k \chi_{ik}(\rho) = \chi_i(\rho) \qquad (i = 1,2,\cdots,\mu). \tag{12}$$

Denote by $D(\rho)$ the determinant of the system (12) and by

$$D_{ij}(\rho) \tag{13}$$

the quotient of the cofactor of the element $\chi_{ij}(\rho)$ in $D(\rho)$ by $D(\rho)$ (supposing of course that $D(\rho) \neq 0$). Then

$$c_j = \sum_{k=1}^{\mu} D_{kj}(\rho)\chi_k(\rho) \qquad (j = 1,2,\cdots,\mu) \tag{14}$$

and substitution in (8) gives

$$\begin{aligned} u(x,\rho) &= \Omega(x,\rho) + \sum_{k,j=1}^{\mu} D_{kj}(\rho)\chi_k(\rho)\Omega_j(x,\rho) \\ &= \int_a^b dtf(t)\left\{G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi \right. \\ &\left. + \sum_{k,j=1}^{\mu} D_{kj}(\rho)\Omega_j(x,\rho)l_k\left(G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi\right)_x\right\}^*. \end{aligned}$$

Hence

$$\begin{aligned} \Gamma(x,t,\rho) &= G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi \\ &+ \sum_{k,j=1}^{\mu} D_{kj}(\rho)\Omega_j(x,\rho)l_k\left(G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi\right)_x. \end{aligned} \tag{15}$$

21. We shall justify these formal operations by proving that, for sufficiently large values of $|\rho|$ in $(\mathfrak{D}_\delta)$, the reciprocal $\mathfrak{H}(x, \xi, \rho)$ exists and the determinant $D(\rho) \neq 0$. By virtue of Theorem 2, then, it will follow that the Green's function $\Gamma(x, t, \rho)$ is meromorphic in ρ, and that the poles of $\Gamma(x, t, \rho)$, for ρ sufficiently large, if there are such, must lie within a distance less than δ from the corresponding poles of $G(x, t, \rho)$. The justification in question is based on the fact that in $(\mathfrak{D}_\delta)$ the function $H(x, \xi, \rho)$ is of order

* The subscript x indicates the independent variable of the operation l_k.

$O(1/\rho)$. This follows immediately from Lemma 1. Let R_δ be chosen so large that for ρ in $(\mathfrak{D}_\delta)$ and $|\rho| \geqq R_\delta$

$$|H(x,\xi,\rho)| \leqq h_0 < 1 .$$

The Neumann series for the reciprocal of the kernel $H(x, \xi, \rho)$, i.e.

$$\mathfrak{H}(x,\xi,\rho) = -\sum_{\nu=0}^{\infty} H_\nu(x,\xi,\rho) ;$$

$$H_0(x,\xi,\rho) = H(x,\xi,\rho) ; \quad H_{\nu+1}(x,\xi,\rho) = \int_a^b H(x,\eta,\rho)H_\nu(\eta,\xi,\rho)d\eta \tag{16}$$

converges uniformly in x, ξ on (a, b) and ρ in $(\mathfrak{D}_\delta)$ and $|\rho| \geqq R_\delta$. In the following $(\mathfrak{D})$ will designate that part of $(\mathfrak{D}_\delta)$ which is outside the circle $|\rho| = R_\delta$.

Expression (16) shows that in $(\mathfrak{D})$

$$\mathfrak{H}(x,t,\rho) = \frac{E}{\rho} . \tag{17}$$

Further we observe that $\Phi_j(x, \rho)$ is a combination of integrals of the type T_1 of Lemma 1, whence an easy application of this Lemma shows, by virtue of conditions (C, i–iv) and formula (17),

$$\Phi_j(x,\rho) = E\rho^{-n+m_j} ; \quad \Omega_j(x,\rho) = E\rho^{-n+m_j},$$
$$l_i(\Omega_k) = \frac{E}{\rho} ; \quad \chi_{ik}(\rho) = [\delta_{ik}] ; \quad \delta_{ik} = \begin{cases} 0 \text{ if } i \neq k, \\ 1 \text{ if } i = k, \end{cases} (i,k,j = 1,2,\cdots,\mu). \tag{18}$$

Hence $D(\rho)$ tends uniformly to 1, as ρ becomes infinite in $(\mathfrak{D})$. Taking R_δ sufficiently large we have then

$$|D(\rho)| \geqq N_\delta > 0 \text{ in } (\mathfrak{D}). \tag{19}$$

Lastly we see that all the $D_{ik}(\rho)$ are bounded in $(\mathfrak{D})$. Thus the following theorem is proved:

THEOREM 3. *Under the conditions* (A), (B), (C), *the Green's function* $\Gamma(x, t, \rho)$ *of the problem* (★), *which is given by* (15), *is meromorphic in* ρ, *and in* $(\mathfrak{D})$ *we have*

$$\Gamma(x,t,\rho) - G(x,t,\rho) = O(1/\rho^n).$$

22. The preceding evaluations enable us to prove the following fundamental proposition:

THEOREM 4. *Under the conditions* (A), (B), (C), *if* (C_R) *denotes a circle* $|\rho| = R$ *in* $(\mathfrak{D})$, *the difference of the two Birkhoff integrals*

$$I_R(f) = \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \Gamma(x,t,\rho)p_{n0}(t)f(t)dt - \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G(x,t,\rho)p_{n0}(t)f(t)dt \tag{20}$$

corresponding to the integro-differential and the differential problems $(\bigstar)$ *tends to zero as* $R\to\infty$, *for any integrable function* $f(x)$, *and uniformly in* x *on* (a, b).

Using the notation of §15 we obviously have

$$I_R(f) = -\frac{1}{2\pi i}\int d\rho \int_a^b \mathfrak{H}(x,\xi,\rho)\boldsymbol{T}_1(f,\xi)d\xi + \frac{1}{2\pi i}\sum_{k,j=1}^{n}\int_{(C_R)} d\rho \boldsymbol{D}_{kj}(\rho)\Omega_j(x,\rho)l_k\Big\{\boldsymbol{T}_1(f,x) - \int_a^b \mathfrak{H}(x,\xi,\rho)\boldsymbol{T}_1(f,\xi)d\xi\Big\}_x . \tag{21}$$

The application of Lemma 1 and the preceding inequalities show that each term under the sign of integral $\int_{(C_R)}$ is of the form* $\epsilon(x,\rho)/\rho$. Hence

$$I_R(f) = \int_{(C_R)} \frac{\epsilon(x,\rho)}{\rho}\rho d \to 0 \text{ as } R\to\infty .$$

Thus, the "equiconvergence theorem" for the integro-differential and differential problems $(\bigstar)$ is established; in other words it is proved that both problems are equivalent in so far as questions of the conditions of convergence, uniform convergence, divergence, or summability by any regular method are concerned.

23. THEOREM 5. *Under the conditions* (A), (B), (C) *the problem* $(\bigstar)$ *possesses infinitely many characteristic values and fundamental functions. If* δ *is any positive number, arbitrarily small, each characteristic value of the problem* $(\bigstar)$, *which is sufficiently large in absolute value, is distant by less than* δ *from a characteristic value of the differential problem* $(\bigstar)$.

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* See third foot note, p. 768.

The second assertion of the theorem has already been proved. To prove the first, let us suppose that the function $\Gamma(x, t, \rho)$ has but a finite number of poles

$$\rho_1, \rho_2, \cdots, \rho_m.$$

Let

$$\Gamma_1(x,t), \Gamma_2(x,t), \cdots, \Gamma_m(x,t)$$

be the corresponding residues of the function $\rho^{n-1}\Gamma(x, t, \rho)$. The integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \Gamma(x,t,\rho)p_{n0}(t)f(t)dt$$

has, then, the same value for all values of R sufficiently large. Hence, in virtue of Theorem 3,

$$\begin{aligned}\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \Gamma(x,t,\rho)p_{n0}(t)f(t)dt \\ = \sum_{\nu=1}^{m} \int_a^b \Gamma_\nu(x,t)p_{n0}(t)f(t)dt \\ = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G(x,t,\rho)p_{n0}(t)f(t)dt = f(x),\end{aligned}$$

$f(x)$ being an arbitrary function satisfying certain conditions of continuity. On setting

$$k(x,t) \equiv \sum_{\nu=1}^{m} \Gamma_\nu(x,t)p_{n0}(t)$$

this relation takes the form of an integral equation

$$f(x) = \int_a^b k(x,t)f(t)dt$$

whose kernel is bounded and integrable and which is possessed of infinitely many linearly independent solutions, which is in contradiction to the classical Fredholm theory.

24. THEOREM 6. *Assuming that the problem* (★) *satisfies the conditions* (A), (B), (C), *let* (★★) *be any differential problem of the type* (★) *which satisfies the conditions* (A) *and* (B) *and whose coefficients* $p_{i0}(x)$, $p_{i1}(x)$, A_{ik}, B_{ik} *and indices* l_{ik} *are the same as those of the problem* (★). *Denote by* $G'(x, t, \rho)$ *the Green's function of the problem* (★★). *Let* $f(x)$ *be any function integrable on* (a, b) *and*

$$(22)\qquad I_R'(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \{\Gamma(x,t,\rho) - G'(x,t,\rho)\}\, p_{n0}(t)f(t)dt.$$

Then, as $R\to\infty$,

$$(23)\qquad I_R'(f) \to \begin{cases} 0 \text{ if } a < x < b, \\ \sum_{i=1}^{n} \int_a^b \{a_i(t) - a_i'(t)\}\Theta_{ai}(t)f(t)dt \text{ if } x = a, \\ \sum_{i=1}^{n} \int_a^b \{a_i(t) - a_i'(t)\}\Theta_{bi}(t)f(t)dt \text{ if } x = b, \end{cases}$$

where $a_i'(x)$ *are the coefficients of the problem* (★★), *which correspond to the coefficients* $a_i(x)$ *of the problem* (★) *and* Θ_{ai}, Θ_{bi} *are known functions which depend merely on the coefficients of the problem* (★) *and do not depend on* $f(x)$. *The convergence of* $I_R'(f)$ *to zero is uniform on every interval interior to* (a, b). *If the functions* $a_i(x)$, $a_i'(x)$ *coincide, then*

$$I_R'(f) \to 0 \text{ as } R \to \infty,$$

uniformly on $(0, 1)$.

It is readily proved that the characteristic values of the differential problem (★★) are asymptotic to those of the differential problem (★) in the sense that if δ is any fixed positive number, arbitrarily small, an R_δ is available which is so large that outside of the circle $|\rho| = R_\delta$ all the characteristic values of (★★) lie in the interiors of the circles of radius δ around the characteristic values of (★) and vice versa (D, 26). Hence the integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G'(x,t,\rho)p_{n0}(t)f(t)dt$$

exists, and Theorem 6 is proved if the assertions of the theorem are proved for the integral

$$j_R(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \{G(x,t,\rho) - G'(x,t,\rho)\}\, p_{n0}(t)f(t)dt.$$

Now, it is readily seen that in the expressions of the Green's functions $G(x, t, \rho)$ and $G'(x, t, \rho)$, as given by Part II, (27), the exponential factors and the leading terms are the same, provided the conditions of Theorem 6 are satisfied. Hence the integral $j_R(f)$ reduces to a sum of terms of the form

$$\int \frac{d\rho}{\rho} \int_a^x e^{\rho r_k(x_k-\xi_k)}[f_1(t)]E(\rho)dt \qquad (k=1,2,\cdots,\tau),$$

(24)

$$\int \frac{d\rho}{\rho} \int_x^b e^{\rho r_k(x_k-\xi_k)}[f_1(t)]E(\rho)dt \qquad (k=\tau+1,\cdots,n);$$

(25) $$\int \frac{\varpi_i'}{\rho}d\rho \int_a^b \varpi_k''[f_1(t)]E(\rho)dt \qquad (i,k=1,2,\cdots,n);$$

(26) $$\int \frac{\varpi_i'}{\rho}d\rho \int_a^b [a_k(t)-a_k'(t)]\Theta_{ik}(t)f(t)dt \qquad (i,k=1,2,\cdots,n).$$

The integration with respect to ρ is taken over those parts of (C_R) which are cut off by the various sectors $(\Re)$ and $f_1(t)$ designates an integrable function. In virtue of Lemma 2, integrals (24) and (25) $\to 0$ as $R\to\infty$, uniformly on (a, b). Integrals (26) are absent if the last condition of Theorem 6 is satisfied. Otherwise integrals (26) $\to\infty$ as $R\to\infty$, uniformly on every interval interior to (a, b), as is shown by

LEMMA 5. *Let (γ) be any part of the circle $|\rho|=r$ which lies in the half-plane* Re $c\rho\leqq 0$ *(c is a constant $\neq 0$). If $\epsilon(\rho, z)$ tends uniformly to zero as r assumes a given set of values tending to ∞, the integral*

$$\int_{(\gamma)} \epsilon(\rho,z)e^{c\rho z}d\rho$$

*tends to zero uniformly on every interval $0<\alpha\leqq z\leqq\beta$.**

Finally, if $x=a$ or $x=b$, integrals (26) obviously tend to limits of the type indicated in (23).

25. COROLLARY. *Let*

$$\rho_1,\rho_2,\cdots,\rho_\nu,\cdots \qquad (|\rho_1|\leqq|\rho_2|\leqq\cdots)$$

be the set of the characteristic values of the problem (★). *Under the conditions* (A), (B), (C) *the series*

$$\sum_{(\nu)} \int_a^b \operatorname{Res}_{\rho=\rho_\nu}\{\rho^{n-1}\Gamma(x,t,\rho)\}p_{n0}(t)f(t)dt$$

converges, for any function $f(x)$ of bounded variation on (a, b), to the values

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} \text{ if } a<x<b,$$

$$A_af(a+0)+B_af(b-0)+\sum_{i=1}^n\int_a^b\Theta_{ai}(t)a_i(t)f(t)dt \text{ if } x=a,$$

$$A_bf(a+0)+B_bf(b-0)+\sum_{i=1}^n\int_a^b\Theta_{bi}(t)a_i(t)f(t)dt \text{ if } x=b,$$

* See D, 38.

where A_a, B_a, A_b, B_b are constants which depend merely on the coefficients of the problem (★) *but not on $f(x)$. The convergence is uniform on every interval of continuity of $f(x)$, which is interior to (a, b).*

This follows immediately from Theorem 6 and from the known properties of the integral*

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G'(x,t,\rho)p_{n0}(t)f(t)dt.$$

Better results concerning the uniformity of convergence on the whole of (a, b) and the values of the series at the end points a, b may be obtained by using another integral in place of the Birkhoff integral. We shall return to this question elsewhere.

26. THEOREM 7. *In addition to* (A), (B), (C) *let us assume that the operator $L(u)$ is of the form*

$$L(u) \equiv u^{(n)} + p_1(x)u^{(n-1)} + \cdots + \{p_n(x) + \rho^n\}u$$

and the operators $A_i(u)+B_i(u)$ $(i=1, 2, \cdots, n)$ are independent of ρ. Suppose furthermore that all the poles of the Green's function $G'(x, t, \rho)$ which are sufficiently large in absolute value are simple.

Then the characteristic values of the integro-differential problem (★) *are asymptotic to those of the differential problem* (★) (*and also to those of any of the problems* (★★)) *in the sense above.*

Let ρ_0 be any one of the poles of the function $G(x, t, \rho)$ (in the region $|\rho| > R_\delta$). Denote by (c) the circle of radius δ around ρ_0. We suppose R_δ so large that all the poles (in the region $|\rho| > R_\delta$) of either of the functions $\Gamma(x, t, \rho)$, $G'(x, t, \rho)$ are interior to the circles (c). To prove that $\Gamma(x, t, \rho)$ has at least one pole in each of the circles (c) it is sufficient to prove that the integral

$$J_c(\Gamma) = \frac{1}{2\pi i}\int_a^b dx \int_{(c)} \Gamma(x, x-0, \rho)d\rho \tag{27}$$

is always different from zero, for if $\Gamma(x, t, \rho)$ has no poles in any one of the circles (c), the integral (27) must vanish. Now, Theorem 3 shows, on the one hand,

$$|J_c(\Gamma - G)| < Ar^{-n},$$

* We may take as the problem (★★) the differential problem whose operators are $L(u)$ and $A_i(u)+B_i(u)$.

where A is a positive constant which depends only on δ, and r denotes the shortest distance from the origin to the contour of (c). On the other hand, we see from Part II, (27), that

$$G(x,t,\rho) - G'(x,t,\rho) = O(r^{-n}) \text{ on } (c),$$
$$|J_c(G - G')| < Ar^{-n}.$$

Hence if we suppose that

$$J_c(\Gamma) = 0,$$

then

$$|J_c(G')| \leqq |J_c(G)| + |J_c(G - G')| = |J_c(\Gamma - G)| + |J_c(G - G')| < 2Ar^{-n}. \tag{28}$$

Now, under the conditions of Theorem 7, it has been proved* that the residue of the function $G'(x, t, \rho)$ corresponding to any simple pole ρ' is

$$\frac{1}{n(\rho')^{n-1}} \sum_{k=1}^{\sigma} u_k(x)v_k(t)$$

where the sets

$$u_1(x), \cdots, u_\sigma(x) \text{ and } v_1(x), v_2(x), \cdots, v_\sigma(x) \tag{29}$$

are composed respectively of the fundamental functions of the problem (★★) and of the adjoint problem (★★) and are biorthogonal and normalized. Supposing that the pole ρ' is within the circle (c) we see that

$$J_c(G') = \frac{\sigma}{n(\rho')^{n-1}},$$

which is in contradiction to (28). We know, however, that no poles of $\Gamma(x, t, \rho)$ (greater than R_δ in absolute value) are outside the circles (c). Hence Theorem 7 is proved.†

27. The method used above can be considerably generalized. It can be immediately applied to the case where the function $h(x, \xi, \rho)$ is replaced by

$$\rho^n h_n(x,\xi) + h(x,\xi,\rho)$$

where $h_n(x, \xi)$ is the sum of a finite number of products of a function of x by a function of ξ. The method can be extended further to the general case where $h_n(x, \xi)$ is any function of the two variables x and ξ, which satisfies certain

* Birkhoff, these Transactions, vol. 9 (1908), pp. 377–380; D, 15.

† It is very probable that Theorem 7 holds true under the more general conditions of Theorem 4. This question, however, requires more detailed knowledge of the structure of the Green's function $G(x, t, \rho)$ in the vicinity of a pole, than that which is available in D.

conditions of continuity and possesses a reciprocal.* This extension (which includes L. Lichtenstein's case) will be treated in another paper by the author. It is sufficient to remark here that Theorem 3 is true in this more general case, but that in Theorem 4, the Birkhoff integral must be replaced by the following one:

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho\left\{\int_a^b f(t)dt\left(\Gamma(x,t,\rho)p_{n0}(t) - \int_a^b \Gamma(x,\xi,\rho)h_n(\xi,t)p_{n0}(\xi)d\xi\right)\right\}.$$

IV. An application to the theory of Fredholm's integral equations†

28. The general theory as indicated above admits of an important application to the theory of Fredholm's integral equations with discontinuous kernels.‡

Let us first develop the formal side of the method; it will be then an easy matter to state all the conditions under which our formal operations are justified.

Consider the integral equation

$$y(t) = \lambda\int_a^\beta k(t,\tau)y(\tau)d\tau \tag{1}$$

whose kernel $k(t, \tau)$ has a finite jump along the line $\tau = t$:

$$k(t,t^+) - k(t,t^-) = k(t,\tau)\Big]_{t^-}^{t^+} = \phi(t) \neq 0.$$

* This condition is essential. For instance, in the example of the foot note on p. 766 all values of ρ are characteristic values of the integro-differential problem (★) while the corresponding differential problem admits only $\rho = 2k\pi$. Another curious example is the following: $u'' + \rho^2 u = \rho^2\int_0^1 \xi u(\xi)d\xi$; $u(0) = 0, \int_0^1 u(\xi)d\xi = 0$. The characteristic values of the integro-differential problem are asymptotic to the roots of the equation $\cos\rho = 1/3$, while those of the differential problem are $\rho = 2k\pi (k = \pm 1, \cdots)$. The characteristic values of the integro-differential problem $u'' + \rho^2 u = 2\rho^2\int_0^1 \xi u(\xi)d\xi$; $u(0) = 0, \int_0^1 u(\xi)d\xi = 0$, are, however, asymptotic to those of the problem

$$u'' + \rho^2 u = 0\,;\ u(0) = 0,\ \int_0^1 u(\xi)d\xi = 0\,.$$

† This chapter was added during the revision of the paper, in May 1927.

‡ An independent treatment of this question has been recently given in the interesting paper by R. E. Langer, *On the theory of integral equations with discontinuous kernels*, these Transactions, vol. 28 (1926), pp. 585–639; in the sequel this paper is denoted by "L." The method which we propose here appears to present some advantage of greater simplicity and generality. We consider here the case only where the kernel itself is discontinuous. The general case of a kernel which is continuous but possesses discontinuous partial derivatives is discussed in a joint paper by Langer and the author.

We assume that, except on the line $\tau = t$, $k(t, \tau)$ is continuous on the square

$$(\mathfrak{S}_0) \qquad \alpha \leqq t \leqq \beta\ ;\ \alpha \leqq \tau \leqq \beta,$$

together with its partial derivatives up to a certain order which will be specified later on, and that all these functions tend to finite limiting values as the point (t, τ) tends to any point on the line $\tau = t$, without crossing this line. It is readily seen that equation (1) can be "normalized" by means of the substitutions (L, Chapter 2)

$$lx = \int_\alpha^t \phi(t)dt\ ;\ l\xi = \int_\alpha^\tau \phi(\tau)d\tau\ ;\ l = \int_\alpha^\beta \phi(t)dt \neq 0\ ;\ \lambda l = \rho\ ;$$

$$\Omega(x,\xi) = k(t,\tau)/\phi(\tau)\ ;\ \psi(x) = \Omega_x(x,\xi)\Big]_{x^+}^{x^-},$$

$$K(x,\xi) \equiv \Omega(x,\xi)e^{\int_\xi^x \psi(x)dx},$$

$$u(x) \equiv y(t)e^{\int_0^x \psi(x)dx}.$$

This brings (1) to the form

$$(2) \qquad u(x) = \rho \int_0^1 K(x,\xi)u(\xi)d\xi.$$

The function $K(x, \xi)$ possesses the same properties of continuity on the square

$$(\mathfrak{S}) \qquad 0 \leqq x \leqq 1\ ;\ 0 \leqq \xi \leqq 1,$$

as the function $k(t, \tau)$ on the square $(\mathfrak{S}_0)$. But, in addition we have now

$$(3) \qquad K(x,\xi)\Big]_{x^-}^{x^+} = 1\ ;\ \ K_x(x,\xi)\Big]_{x^-}^{x^+} = 0.$$

So far the function $K_x(x, \xi)$ is defined for $x \neq \xi$. On setting

$$K_x(x,\xi)\big|_{\xi=x} = K_x(x,x^-) = K_x(x,x^+)$$

we obtain a function which is defined and continuous on the whole of the square $(\mathfrak{S})$.

From now on we shall deal exclusively with the normalized equation (2)*.

29. If we differentiate (2) we get, on account of (3),

$$(4) \qquad u(x) = -u'(x)/\rho + \int_0^1 K_x(x,\xi)u(\xi)d\xi.$$

* The normalization of (1) is not at all necessary for our purposes; we might discuss (1) without any transformation as well. It is only for the sake of simplicity of computations that we prefer to deal with the normalized equation (2).

Here we consider $-u'(x)/\rho$ as a known function and assume that the kernel $K_x(x, \xi)$ possesses a reciprocal, $\mathfrak{E}(x, \xi)$, which is defined by

$$(5) \qquad K_x(x,\xi) + \mathfrak{E}(x,\xi) = \int_0^1 K_x(x,s)\,\mathfrak{E}(s,\xi)ds = \int_0^1 \mathfrak{E}(x,s)K_s(s,\xi)ds.$$

The function $\mathfrak{E}(x, \xi)$ has on $(\mathfrak{S})$ the same properties of continuity as the function $K_x(x, \xi)$; in particular, $\mathfrak{E}(x, \xi)$ is continuous on $(\mathfrak{S})$. We see at once that

$$\rho u(x) = -u'(x) + \int_0^1 \mathfrak{E}(x,\xi)u'(\xi)d\xi,$$

and an integration by parts gives the integro-differential equation which is satisfied by $u(x)$:

$$(6) \qquad u'(x) + \rho u(x) = u(1)\,\mathfrak{E}(x,1) - u(0)\,\mathfrak{E}(x,0) - \int_0^1 \mathfrak{E}_\xi(x,\xi)u(\xi)d\xi\,.$$

It remains now to find the boundary condition which is satisfied by $u(x)$. This can be done by substituting $x=0$ into (2):

$$u(0) = \rho \int_0^1 K(0,\xi)u(\xi)d\xi.$$

The parameter ρ may be eliminated from here by means of (6). If we substitute $\rho u(x)$ from (6) and integrate by parts, we easily find

$$(7) \qquad au(0) + bu(1) = \int_0^1 \alpha(\xi)u(\xi)d\xi$$

where

$$a = 1 - K(0,0^+) + \int_0^1 K(0,\xi)\,\mathfrak{E}(\xi,0)d\xi,$$

$$(8) \qquad b = K(0,1) - \int_0^1 K(0,\xi)\,\mathfrak{E}(\xi,1)d\xi,$$

$$\alpha(\xi) = K_\xi(0,\xi) - \int_0^1 K(0,s)\,\mathfrak{E}_\xi(s,\xi)ds.$$

Thus we have proved that every solution of (2) is a solution of the homogeneous integro-differential problem $(\bigstar)$ consisting of (6) and (7). The complete equivalence of those two problems will be proved if we can show that, conversely, every solution of (6), (7) is a solution of (2).

where

$$(11) \qquad F(x) = -\int_0^1 K(x,\xi)f(\xi)d\xi.$$

On setting $f(x)\equiv 0$, we obtain the desired proof of the equivalence of the integral equation (2) and of the homogeneous integro-differential problem $(\bigstar)$.

31. Using the results of §30 we can establish the important fact that the Green's function $\Gamma(x, t, \rho)$ of the integro-differential problem $(\bigstar)$ and the resolvent kernel $\mathfrak{K}(x, t, \rho)$ of (2) are identical. The function $\rho\mathfrak{K}(x, t, \rho)$ is defined as the reciprocal of the kernel $\rho K(x, \xi)$ so that*

$$K(x,\xi) + \mathfrak{K}(x,\xi,\rho) = \rho \int_0^1 K(x,s)\mathfrak{K}(s,\xi,\rho)ds = \rho \int_0^1 \mathfrak{K}(x,s,\rho)K(s,\xi)ds.$$

The homogeneous problems (2) and $(\bigstar)$ being equivalent, they have the same set of characteristic values, say

$$\rho_1, \rho_2, \rho_3, \cdots, \rho_\nu, \cdots.$$

Let ρ be different from any ρ_ν. The solution of the non-homogeneous problem $(\bigstar)$, then, is uniquely determined and given by

$$u(x) = \int_0^1 \Gamma(x,t,\rho)f(t)dt.$$

On the other hand, from (10), (11) we have

$$\begin{aligned} u(x) &= F(x) - \rho \int_0^1 \mathfrak{K}(x,\xi,\rho)F(\xi)d\xi \\ &= \int_0^1 f(t)dt\left\{ -K(x,t) + \rho \int_0^1 \mathfrak{K}(x,\xi,\rho)K(\xi,t)d\xi \right\} \\ &= \int_0^1 \mathfrak{K}(x,t,\rho)f(t)dt, \end{aligned}$$

which, in view of the arbitrariness of $f(x)$, shows that

$$\Gamma(x,t,\rho) \equiv \mathfrak{K}(x,t,\rho).$$

This being stated, the integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_0^1 \Gamma(x,t,\rho)p_{n0}(t)f(t)dt$$

* Cf. Lalesco, *Introduction à la Théorie des Equations Intégrales*, Paris, 1912, pp. 23–24. It should be noted that our kernel $K(x, \xi)$ corresponds to the kernel $-N(x, \xi)$ of Lalesco.

30. It is desirable to answer a more general question, namely, what is the integral equation which is satisfied by a solution of the system consisting of the *non-homogeneous* integro-differential equation

$$(9)\quad u'(x) + \rho u(x) = f(x) + u(1)\,\mathfrak{E}(x,1) - u(0)\,\mathfrak{E}(x,0) - \int_0^1 \mathfrak{E}_\xi(x,\xi)u(\xi)d\xi,$$

and the boundary condition (7), $f(x)$ being an arbitrary continuous function.

First, we have from (9)

$$u'(x) = f(x) - \rho u(x) + \int_0^1 \mathfrak{E}(x,\xi)u'(\xi)d\xi,$$

whence, $K_x(x, \xi)$ being the reciprocal of $\mathfrak{E}(x, \xi)$,

$$u'(x) = f(x) - \rho u(x) - \int_0^1 K_x(x,\xi)\{f(\xi) - \rho u(\xi)\}d\xi,$$

or else

$$u'(x) = f(x) - \int_0^1 K_x(x,\xi)f(\xi)d\xi + \rho\left\{ - u(x) + \int_0^1 K_x(x,\xi)u(\xi)d\xi \right\}$$

$$= \frac{d}{dx}\left\{ \rho \int_0^1 K(x,\xi)u(\xi)d\xi \right\} - \frac{d}{dx}\left\{ \int_0^1 K(x,\xi)f(\xi)d\xi \right\},$$

and finally

$$u(x) = - \int_0^1 K(x,\xi)f(\xi)d\xi + \rho \int_0^1 K(x,\xi)u(\xi)d\xi + C,$$

where C is a constant.

Now, to find C, we rewrite (7) as follows:

$$u(0) = - \int_0^1 K(0,\xi)u'(\xi)d\xi - \int_0^1 u(\xi)d\xi \int_0^1 K(0,s)\,\mathfrak{E}_\xi(s,\xi)ds$$

$$+ u(1) \int_0^1 K(0,\xi)\,\mathfrak{E}(\xi,1)d\xi - u(0) \int_0^1 K(0,\xi)\,\mathfrak{E}(\xi,0)d\xi,$$

and, substituting $-u'(\xi)$ from (9),

$$u(0) = \rho \int_0^1 K(0,\xi)u(\xi)d\xi - \int_0^1 K(0,\xi)f(\xi)d\xi.$$

Hence, $C = 0$ and $u(x)$ is a solution of the non-homogeneous integral equation

$$(10)\qquad u(x) = F(x) + \rho \int_0^1 K(x,\xi)u(\xi)d\xi$$

used in Part III reduces in the present case to

$$(12)\qquad \frac{1}{2\pi i}\int_{(C_R)} d\rho \int_0^1 \mathfrak{K}(x,t,\rho)f(t)dt = \sum_{\nu=1}^{N} \int_0^1 f(t) \operatorname*{Res}_{\rho=\rho_\nu} \mathfrak{K}(x,t,\rho)dt$$

where $\rho_1, \rho_2, \cdots, \rho_N$ are the characteristic values of (2) within (C_R). It is well known, however, that the principal part of $\mathfrak{K}(x, t, \rho)$ corresponding to a pole ρ_0, of multiplicity m, is

$$\frac{\phi_m(x,t)}{(\rho - \rho_0)^m} + \cdots + \frac{\phi_1(x,t)}{\rho - \rho_0}$$

where

$$\phi_1(x,t) = \sum_{(i)} \Phi_i(x)\Psi_i(t)$$

and

$$\Phi_1(x), \cdots; \ \Psi_1(x), \cdots$$

are respectively the *fundamental functions* of the equation (2) and of the associated equation

$$(13)\qquad v(x) = \rho \int_0^1 K(\xi,x)v(\xi)d\xi$$

corresponding to the pole ρ_0.* The functions $\Phi_1(x)$, $\Psi_1(x)$, $\cdots$ constitute a biorthogonal system of function which reduces to that of the *fundamental solutions* of (2) and (13), when and only when the pole ρ_0 is simple. Now, let

$$u_1(x), v_1(x) ; u_2(x), v_2(x) ; \cdots, u_\nu(x), v_\nu(x), \cdots$$

be the complete set of fundamental functions of the equations (2) and (13) which, as we know, is biorthogonal. Let ρ_ν denote the pole of $\mathfrak{K}(x, t, \rho)$ which corresponds to the pair of functions $u_\nu(x), v_\nu(x)$, each pole being counted as many times as there are linearly independent fundamental functions corresponding to this pole, and

$$\sum (f) \equiv \sum_{\nu=1}^{\infty} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt.$$

The integral

$$\Im(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} d\rho \int_0^1 \Gamma(x,t,\rho)f(t)d\rho$$

becomes then

$$\sum_{(C_R)} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt \equiv \sum_{(C_R)} (f)$$

* Lalesco, loc. cit., pp. 46–55.